NONTANGENTIAL SUMMABILITY OF CONJUGATE FOURIER SERIES

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We consider the linear means of the conjugate Fourier series of integrable 2π-periodic function $f(y)$, generated by summing infinite sequence $\lambda(h)$. For the positive values of h, and any x, the behavior of λ-means is investigated, when a point (y, h) tends to $(x, 0)$ within the fixed “corner” area $\Gamma(x)$. In the case of the summation sequences, decreasing quickly enough, the estimates of strong and weak type of corresponding maximal operators are obtained. We establish the convergence of λ-means to the conjugate function, when (y, h) tends to $(x, 0)$ along the paths within $\Gamma(x)$. An important special case of received statements is a non-tangential convergence of λ-means for summation methods of exponential type. Results include classical case of Poisson-Abel means.

Keywords: conjugate series; estimates of the weak and strong type; non-tangential summability

Formulation of the problem

Denote $Q = [-\pi, \pi]$; let $L^p = L^p(Q)$ be Lebesgue class of 2π-periodic functions of a real variable, for which

$$\|f\|_p = \left(\int_Q |f(x)|^p\, dx\right)^{1/p} < \infty,\ p \ge 1,$$

set $L = L(Q) = L^1(Q)$. Let

$$\lambda = \{\lambda_k(h),\ k = 0, 1, ...;\ \lambda_0(h) = 1\} \qquad (1)$$

be an arbitrary sequence infinite, generally speaking, determined by values of parameter $h > 0$. In this paper we study the behavior of λ-means

$$\tilde{U}_h(f) = \tilde{U}(f, y; \lambda, h) = -i\sum_{k=-\infty}^{\infty} (\operatorname{sgn} k)\lambda_{|k|}(h)c_k(f)\exp(iky), \qquad (2)$$

of conjugate Fourier series

$$-i\sum_{k=-\infty}^{\infty} (\operatorname{sgn} k)c_k(f)\exp(iky);\quad c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\exp(-ikt)\,dt,\quad k = 0, \pm1, \pm2, ...; \qquad (3)$$

when $(y, h) \to (x, +0)$ along the paths within

$$\Gamma_d(x) = \left\{(y,h)\,\middle|\, y\in[-\pi, \pi],\ 0<h<1,\ \frac{|y-x|}{h}\le d\right\},\quad d = \text{const},\quad d>0$$

(tending along non-tangential paths). We generalize and strengthen some of the results of [3, 4, 5].

Maximal operators

Denote

$$f^* = f^*(x) = \sup_{\eta>0}\frac{1}{\eta}\int_{x-\eta}^{x+\eta}|f(t)|\,dt;\quad \tilde{f}^* = \tilde{f}^*(x) = \sup_{\eta>0}\left|\int_{\eta\le|t|\le\pi}\frac{f(x+t)}{2\operatorname{tg}\frac{t}{2}}dt\right|; \qquad (3)$$

f^* and $\tilde{f}^*$ are defined ([1], vol. 1, p. 60, 401–402, 442, 443) for every $f \in L$; moreover, in this case there is almost everywhere a conjugate function

$$\tilde{f}(x) = -\frac{1}{\pi}\lim_{\eta\to+0}\int_{\eta\le|t|\le\pi} f(x+t)\operatorname{ctg}\frac{t}{2}\,dt.$$

In accordance with λ-means (2), introduced above, we define maximal operator

$$\tilde{U}_*(f) = \tilde{U}_*(f, x; \lambda) = \sup_{(y,\,h)\in\Gamma_d(x)}\left|\tilde{U}(f, x; \lambda, h)\right|. \qquad (4)$$

For each $h > 0$ denote $m = \left[\frac{1}{2dh}\right]$. The basis of the results of the behavior of means (2) is the following statement.

Theorem 1. Let the sequence (1) decreases so rapidly that

$$N\left|\lambda_N(h)\right| + N^2\left|\Delta\lambda_N(h)\right| = o(1), \quad N \to \infty, \quad (5)$$

and

$$\sum_{k=1}^{\infty} \frac{k(k+m)}{m}\left|\Delta^2\lambda_k(h)\right| \le C_\lambda. \quad (6)$$

Then, for all $f \in L(Q)$ the estimate

$$\tilde{U}_*(f, x; \lambda) \le C_\lambda\left(f^*(x) + \tilde{f}^*(x)\right) \quad (7)$$

holds.

Here and below C will represent constants, which depend only on clearly specified indexes.

Auxiliary assertion

Consider ([2], vol. 1, pp. 86, 153) the conjugate Dirichlet kernel

$$\tilde{D}_k(t) = \sum_{\nu=1}^{k} \sin \nu t = \frac{1}{2\operatorname{tg}\frac{1}{2}t} - \frac{\cos\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}$$

and the conjugate Fejer kernel

$$\tilde{F}_k(t) = \frac{1}{k+1}\sum_{\nu=0}^{k} \tilde{D}(t) = \frac{1}{2\operatorname{tg}\frac{1}{2}t} - \tilde{\tilde{F}}_k(t), \quad (8)$$

where $\tilde{\tilde{F}}_k(t) = \dfrac{\cos(k+1)t}{2(k+1)\sin^2\frac{1}{2}t}$; $k = 0, 1, \ldots$;

$\tilde{D}_0(t) = \tilde{F}_{-1}(t) = 0.$

Lemma. For all $k = 0, 1, \ldots$ and $(y, h) \in \Gamma_d(x)$ the estimate

$$\left|\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)dt\right| \le C\left(1+\frac{k}{m}\right)\left(f^*(x) + \tilde{f}^*(x)\right) \quad (9)$$

holds.

Proof. Let's start with a few comments. At $k = 0$ the left side of (9) vanishes, so consider $k = 1, 2, \ldots$

If $(y, h) \in \Gamma_d(x)$, then, obviously, $|y-t| \ge |x-t| - dh$. Hence, for x and t, such, that $|x-t| \ge \frac{1}{m} \ge 2dh$, the estimate

$$|y-t| \ge \frac{1}{2}|x-t| \quad (10)$$

is valid. Indeed, (10) follows from inequality $|y-t| \ge |x-t| - dh \ge \frac{1}{2}|x-t|$ for all $(y, h) \in \Gamma_d(x)$. Then, by definitions (8), the estimates

$$\left|\tilde{F}_k(t)\right| \le Ck; \quad |t| \le \pi;$$

$$\left|\tilde{\tilde{F}}_k(t)\right| \le C\frac{1}{kt^2}; \quad 0 < |t| \le \pi \quad (11)$$

hold.

Assume firstly $k \le m$ and obtain the relation (9). By (11) we have

$$\left|\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)dt\right| = \left|\int_{x-\pi}^{x+\pi} f(t)\tilde{F}_k(y-t)dt\right| = \left|\int_{|x-t|\le\pi} f(t)\tilde{F}_k(y-t)dt\right| \le C\Bigg(k\int_{|x-t|\le\frac{1}{k}}|f(t)|dt +$$

$$+\left|\int_{\frac{1}{k}\le|x-t|\le\pi} f(t)\operatorname{ctg}\frac{y-t}{2}dt\right| + \int_{\frac{1}{k}\le|x-t|\le\pi}|f(t)|\cdot\left|\tilde{\tilde{F}}_k(y-t)\right|dt = \quad (12)$$

$$= C\left(J_1(x,k) + J_2(x,k) + J_3(x,k)\right).$$

It is obvious that

$$J_1(x, k) \le f^*(x). \quad (13)$$

Further,

$$J_2(x,k) = \left|\int_{\frac{1}{k}\le|x-t|\le\pi} f(t)\operatorname{ctg}\frac{x-t}{2}dt + \int_{\frac{1}{k}\le|x-t|\le\pi} f(t)\frac{\sin\frac{x-y}{2}}{\sin\frac{x-t}{2}\sin\frac{y-t}{2}}dt\right|.$$

Taking into account (10), we have

$$|J_2(x,k)| \le C\left(\tilde{f}^*(x+h)\int_{\frac{1}{k}\le|x-t|\le\pi}|f(t)|\frac{1}{(x-t)^2}dt\right).$$

Here

$$\int_{\frac{1}{k}\le|x-t|\le\pi}|f(t)|\frac{1}{(x-t)^2}dt \le Ck\sum_{j=1}^{S}\frac{k}{(2^{j-1})^2}\int_{\frac{2^{j-1}}{k}\le t\le\frac{2^j}{k}}|f(x+t)|dt \le Ck\, f^*(x),$$

if a positive integer S chosen from the condition

$$\frac{2^{S-1}}{k}\le\pi<\frac{2^S}{k}.$$

Hence

$$|J_2(x,k)| \le C\left(\tilde{f}^*(x)+\frac{k}{m}f^*(x)\right) \le C\left(\tilde{f}^*(x)+f^*(x)\right). \quad (14)$$

Finally, in view of (10) and (11)

$$J_3(x,k) \le Cf^*(x). \quad (15)$$

Now, according to (12)–(15), the estimate (9) is valid at all $k \le m$.
Consider now the case of $k > m$. By (11) we have

$$\left|\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)\,dt\right| \le C\left(\int_{|x-t|\le 1/m}|f(t)|k\,dt + \left|\int_{\frac{1}{m}\le|x-t|\le\pi} f(t)\,\mathrm{ctg}\frac{y-t}{2}dt\right| + \right.$$

$$\left. + \int_{\frac{1}{m}\le|x-t|\le\pi}|f(t)|\left|\tilde{\tilde{F}}_k(y-t)\right|dt\right) = C\left(\frac{k}{m}J_1(x,m)+J_2(x,m)+\mathrm{I}(x,k,m)\right). \quad (16)$$

According to (13) and (14) we obtain

$$J_1(x,m)\le f^*(x); \quad |J_2(x,m)| \le C\left(\tilde{f}^*(x)+f^*(x)\right).$$

Further, in view of (11) and (10)

$$\mathrm{I}(x,k,m) \le C\frac{m}{k}f^*(x) \le C\,f^*(x).$$

It follows now from (16) that

$$\left|\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)\,dt\right| \le C\left(1+\frac{k}{m}\right)\left(\tilde{f}^*(x)+f^*(x)\right)$$

for all $k > m$.

Thus, the estimate (9) is valid for all $k = 1,2, \ldots$, and lemma is proved.

Proof of Theorem 1

Applying (3), Abel transform twice ([2], vol. 1, p. 15), the obvious estimate $\left|\tilde{D}_N(t)\right| \le N$, $N = 1, 2, \ldots$, and (11), we obtain for (2)

$$\left|\tilde{U}(f, y; \lambda, h)\right| = \left|\lim_{N\to+\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\sum_{k=1}^{N} \lambda_k(h)\sin k(y-t)\right\}dt\right| =$$

$$= \frac{1}{\pi}\left|\lim_{N\to+\infty}\left\{\lambda_N(h)\int_{-\pi}^{\pi} f(t)\tilde{D}_N(y-t)\,dt + N\Delta\lambda_{N-1}(h)\int_{-\pi}^{\pi} f(t)\tilde{F}_{N-1}(y-t)\,dt + \right.\right.$$

$$\left.\left. + \sum_{k=1}^{N-2}(k+1)\Delta^2\lambda_k(h)\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)\,dt\right\}\right| \le C\lim_{N\to+\infty}\left\{\left(N\left|\lambda_N(h)\right| + \right.\right.$$

$$\left.\left. + N^2\left|\Delta\lambda_N(h)\right|\right)\int_{-\pi}^{\pi}\left|f(t)\right|dt + \sum_{k=1}^{N-2}(k+1)\left|\Delta^2\lambda_k(h)\right|\cdot\left|\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)\,dt\right|\right\}.$$

According to (5) and (9) we have

$$\left|\tilde{U}(f, y; \lambda, h)\right| \le C\sum_{k=1}^{\infty}(k+1)\left|\Delta^2\lambda_k(h)\right|\cdot\left|\int_{-\pi}^{\pi} f(t)\tilde{F}_k(y-t)\,dt\right| \le$$

$$\le C\left(\tilde{f}^*(x) + f^*(x)\right)\sum_{k=1}^{\infty}\left|\Delta^2\lambda_k(h)\right|\cdot k\left(1 + \frac{k}{m}\right),$$

and, because of the condition (19), we obtain the assertion (7).

Estimates of the weak and strong type

Theorem 2. Under the conditions of Theorem 1 the estimates of weak type

$$\mu\left\{x \in Q \middle| \tilde{U}_*(f, x; \lambda) > \varsigma > 0\right\} \le C_{p,\lambda}\left(\frac{\|f\|_p}{\varsigma}\right)^p, \quad p \ge 1$$

and strong type

$$\left\|\tilde{U}_*(f)\right\|_p \le C_{p,\lambda}\|f\|_p, \quad p > 1;$$

$$\left\|\tilde{U}_*(f)\right\| \le C_\lambda\left(1 + \left\|f\left(\ln^+|f|\right)\right\|\right);$$

$$\left\|\tilde{U}_*(f)\right\|_p \le C_{p,\lambda}\|f\|, \quad 0 < p < 1$$

are valid.

The assertion follows from Theorem 1 and the corresponding estimates of weak and strong type for (3); see ([2], vol.1, pp. 58–59, 404).

Non-tangential summability

Theorem 3. If the sequence (1) satisfies to the conditions (5), (6) and

$$\lim_{h\to 0}\lambda_k(h) = 1, \quad k = 0, 1, \ldots, \qquad (17)$$

then the relation

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}} \tilde{U}(f, y; \lambda, h) = \tilde{f}(x) \qquad (18)$$

holds almost everywhere for each $f \in L(Q)$.

The relations (18) follows from the weak type estimates (theorem 2) and condition (17) by the standard method ([2], vol. 2, pp. 464–465).

Piecewise convex summation methods

It noted in [3–5] (cf. [2], p. 476–478) that under the condition (5) every piecewise-convex sequence (1) satisfies the condition

$$\sum_{k=1}^{\infty} k\left|\Delta^2\lambda_k(h)\right| \le C_\lambda.$$

By virtue of piecewise convexity of sequence (1), the second finite differences $\Delta^2\lambda_k(h)$ retain the sign; suppose for definiteness, it will be a plus sign at all sufficiently large k (depending, generally speaking, from h), namely $k \ge \tau(m)$, where $\tau = \tau(m)$ – some positive integer,

$$\tau = \tau(m) = \tau(m, \lambda) \le m. \qquad (20)$$

The sum (6) does not exceed

$$C_\lambda\left(\sum_{k=1}^{\infty} k\left|\Delta^2\lambda_k(h)\right| + \sum_{k=m}^{\infty}\tfrac{k^2}{m}\left|\Delta^2\lambda_k(h)\right|\right). \quad (21)$$

In the second sum of (21) all $\Delta^2\lambda_k(h)$ are positive by (20); applying twice Abel transform, we have

$$\sum_{k=m}^{\infty}\left|\Delta^2\lambda_k(h)\right|\frac{k^2}{m} = \frac{1}{m}\sum_{k=m}^{\infty}k^2\cdot\Delta^2\lambda_k(h) = \frac{1}{m}\left(m^2\Delta^2\lambda_m(h) + k\sum_{k=m+1}^{\infty}(2\quad -1)\cdot\Delta\ \lambda_k(h)\right) =$$

$$= m\Delta^2\lambda_m(h) + \frac{2m+1}{m}\Delta\lambda_{m+1}(h) + \frac{1}{m}\sum_{k=m+2}^{\infty}\lambda_k(h).$$

Thus, under conditions (5) and

$$\frac{1}{m}\sum_{k=m+2}^{\infty}\left|\lambda_k(h)\right| \le C_\lambda, \quad (22)$$

the assertions of Theorems 2 and 3 are valid for each piecewise-convex sequence (1).

Exponential summation methods

Summation methods

$$\lambda_0(h) = 1, \qquad \lambda_k(h) = \lambda(x, h)\big|_{x=k}, \quad k = 1, 2, \ldots, \text{ where } \lambda(x, h) = \exp(-h\varphi(x))$$

were studied in [3–4] in the case of "radial" convergence; in particular, it was given the condition of piecewise convexity of sequence $\{\lambda_k(h)\}$ In this paper we consider

$$\lambda(x, h) = \exp(-hx^\alpha), \ \alpha \ge 1$$

It is easy to show that this function is a piecewise-convex; verify now the satisfialibity of condition (22). We have

$$\frac{1}{m}\sum_{k=m+2}^{\infty}\left|\lambda_k(h)\right| \le \frac{1}{m}\int_0^{\infty}\exp(-hx^\alpha)dx = \frac{1}{\alpha m}h^{-\frac{1}{\alpha}}\int_0^{\infty}t^{\frac{1}{\alpha}-1}\exp(-t)dt \le C_\alpha h^{1-\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}\right), \quad (23)$$

where $\Gamma = \Gamma\left(\frac{1}{\alpha}\right)$ is Euler gamma function. For $\alpha \ge 1$ the right side of (23) does not exceed a constant that depends only on α. Thus, Theorems 2 and 3 are valid for exponential summation methods $\lambda_k(h) = \exp(-hk^\alpha)$, $\alpha \ge 1$; for $\alpha = 1$ we have classical Poisson-Abel means.

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