ON THE NEW NON-HAMILTONIAN QUATERNIONS OF HALF-ROTATION AND THEIR APPLICATION TO PROBLEMS OF ORIENTATION

[1]Panov A., [2]Ponomarenko S.

[1]International Academy of Navigation and Motion Control, Ukrainian Department, Kyiv, e-mail: anatoliy_panov@ukr.net;

[2]State Research Institute of Aviation, Kyiv, e-mail: sol_@ukr.net

The application new (non-traditional) groups, algebras of the half-rotation solid body and their application in the orientation tasks for the strapdown inertial navigation system and orientation systems. Non-Hamiltonian quaternions of the half-rotation can be zero in contrast to the classical Hamiltonian normalized quaternions of the full-rotation with the parameters of the Euler (Rodrigues – Hamilton), their rates are not constant and depend on the Euler angles of final rotation.

Keywords: non-Hamiltonian quaternion of rotation, half-rotation, groups and algebras of quaternions, strapdown inertial systems, guidance, navigation, control

In strapdown inertial navigation system (SINS) [1] and orientation systems (SIOS) [9] of the aerospace aerial vehicle the classical "Hamiltonian" quaternions of solid body rotation with the parameters of the Euler (Rodrigues – Hamilton) [1], [9] are now (from the beginning of the 70s of the last century) widely used. These quaternions are normalized (with unit norm) and they cannot be zero [1–10].

The possibility of using an unnormalized quaternion for SINS with no single the norms, depending on the angle of the final Euler rotation of solid body first is shown in [5 (2000), 10 (1999)]. Such quaternions are obtained by multiplying the normalized Hamiltonian quaternions of rotation (as unit vectors of the real four-dimensional space) by an arbitrary function of the angle of Euler rotation. They belong to the sets of *non-Hamiltonian quaternions* of the solid body *"full" rotation*.

The paper examines the new (previously published in [6]) unnormalized quaternions of rotation forming a set of *non-Hamiltonian quaternions* of the solid body *"half-rotation"*.

Non-Hamiltonian unnormalized quaternions of half-rotation are exceptional by virtue of their properties, in particular, the heterogeneity of systems of four kinematic linear differential equations corresponding to these quaternions.

Non-Hamiltonian quaternions of the half-rotation

We are considering two types of non-Hamiltonian, quaternions of the half-rotation of solid body:

$$U = u_0 + \bar{\lambda}; \quad V = v_0 + \bar{\lambda},$$

where $u_0 = 1 - \lambda_0$; $v_0 = 1 + \lambda_0$; $\lambda_0 = \cos(\varphi/2)$; $\bar{\lambda} = \lambda\bar{k}$; $\lambda = \sin(\varphi/2)$; $\bar{k}$ is the unit vector of Euler's axis of finite rotation (turn) of the solid body in three-dimensional Euclidean vector space [1; 3; 9]; φ is the Euler final rotation angle.

Parameter λ_0 and coordinates λ_n (n = 1, 2, 3) of three dimensional vector $\bar{\lambda}$ (coordinate orthonormal basis with unit vectors related to a solid body) are Euler (Rodrigues – Hamilton as a function of the angle φ) real parameters [1; 3; 9; 10]. They define the classic Hamiltonian quaternion of "full" rotation [1; 3]:

$$\Lambda = \lambda_0 + \bar{\lambda}$$

with unit norm

$$\|\Lambda\| = \lambda_0^2 + \lambda^2 = 1; \quad \lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

Quaternions U, V are considered here as *non-Hamiltonian quaternions of half-rotation* of solid body and turn out as a result of multiplication of non-traditional new *normalized quaternion of half-rotation*

$$P = m + \bar{p}; \quad M = p + \bar{m}; \quad m = \sin(\varphi/4);$$

$$\bar{p} = \cos(\varphi/4)\bar{k}; \quad p = \cos(\varphi/4);$$

$$\bar{m} = \sin(\varphi/4)\bar{k}$$

respectively on the modules

$$|U| = 2m; \quad |V| = 2p,$$

(i.e. $U = |U|P$, $V = |V|M$). This normalized "Hamiltonian" quaternions of half-rotation P, M are regarded as vectors in the real four-dimensional vector space.

The different sets of the half-rotation quaternions determined by the *generalized non-Hamiltonian quaternions of the half-rotation* $U_C = c_U U$; $V_C = c_V V$, where c_U, c_V are arbitrary constant coefficients. When $c_U = c_V = 1$ viewed quaternions U, V are obtained.

Unlike quaternions Λ, unnormalized quaternions U, V can be zero (at $\varphi = 0$ and $\varphi = 2\pi$

respectively) and their modules depend on angle φ. Therefore, they are of special practical interest in solving two major problems: inertial sensing and inertial attitude control of the solid body provided that the shortest turns (at angles $\varphi < \pi$ and $\varphi > \pi$) are ensured.

Quaternions U, V are exceptional (from the set of possible non-Hamiltonian unnormalized quaternions of rotation [1; 5; 6; 10]) as those quaternions and their corresponding kinematic differential equations and groups, group quaternions algebras of rotation have a number of special or unique properties.

By the way for example, quaternions U, V in addition to going to zero, have a common vector $\bar{\lambda}$, and their norms are equal to doubled scalar parts:

$$\|U\| = 2u_0 = U \circ \tilde{U} = u_0^2 + \lambda^2;$$

$$\|V\| = 2v_0 = V \circ \tilde{V} = v_0^2 + \lambda^2, \qquad (1)$$

where $\tilde{U} = (u_0 - \bar{\lambda})$; $\tilde{V} = (v_0 - \bar{\lambda})$ are conjugate quaternions.

In addition, the following equalities hold: $u_0 v_0 = \lambda^2 = (\bar{\lambda} \cdot \bar{\lambda})$, and $U + \tilde{U} = U \circ \tilde{U}$, $V + \tilde{V} = V \circ \tilde{V}$, unlike inequality $\Lambda + \tilde{\Lambda} \neq \Lambda \circ \tilde{\Lambda}$, where (°) is the sign algebraic operations "Hamiltonian" quaternion multiplication [1; 3].

Quaternion differential kinematic equations

Quaternion kinematic differential equations for "proper" quaternions [1; 9, p. 109] U, V, are linear, but not homogeneous. Those equations are obtained from the known [1] linear kinematic equations $2\dot{\Lambda} = \Lambda$ for quaternion Λ by substitution of variable λ_0 with variables u_0, v_0 and are as follows:

$$2\dot{U} = \Omega - \Omega \circ U; \quad 2\dot{V} = -\Omega + V \circ \Omega, \qquad (2)$$

where $\Omega = (0 + \bar{\omega})$ is the angular velocity quaternion; $\bar{\omega}$ is the vector of absolute rotational velocity of the solid body; $\dot{\Lambda}$, $\dot{U}$, $\dot{V}$ is the relative derivatives of quaternions in time.

The equations (2) have a joint first integral $u_0 + v_0 = 2$.

These equations because of their inhomogeneity are of special interest for the solution of tasks of synthesis of high-precision conical precession computer algorithms of SIOS (the sixth or tenth order of accuracy) using Taylor's rows [9].

The formulas for the multiplication of quaternions of the half-rotation

The multiplication formulas (rules, laws) [9, p. 109] of proper non-Hamiltonian quaternions U, V, are obtained from the classic (group) [1; 3] multiplication formulas of normalized own quaternions Λ by substitution of quaternion Λ with quaternions U, V, according to the following formulas

$$\Lambda = E_4 - \tilde{U} = V + E_4,$$

where $E_4 = (1 + \bar{o})$ is a scalar unit quaternion; $\bar{o}$ zero vector.

For two sequential finite rotations (turns) of the solid body, the group multiplication formulas of normalized quaternions Λ and non-Hamiltonian quaternions U, V are written in symbolic form as:

$$\Lambda = \Lambda_1 \circ \Lambda_2; \quad U = U_1 \otimes U_2;$$

$$V = V_1 \otimes V_2,$$

as well as:

$$U = U_1 + U_2 - U_2 \circ U_1;$$

$$V = 2E_4 - V_1 - V_2 + V_1 \circ V_2, \qquad (3)$$

where Λ, U, V are the resulting rotation quaternions, Λ_1, U_1, V_1 are the first rotation quaternions, Λ_2, U_2, V_2 are the second rotation quaternions; (⊗) is a conventional sign of the group (non-Hamiltonian) multiplication [6; 10] of any non-normalized quaternions; (°) is a sign of the algebraic operation of Hamiltonian multiplication.

The formula (3) includes the operation of addition of quaternions, in contrast of the multiplication formulas of the classical Hamiltonian quaternions with the parameters of the Euler [1; 3; [4].

The group of non-Hamiltonian quaternions of the half-rotation

The quaternion sets Λ, U, V, form a four-dimensional quaternions representations of three-dimensional rotations classical groups [3; 4; 6] – *a groups of non-Hamiltonian quaternions of three-dimensional rotations and half-rotation* of the solid body or of quaternion groups of three-dimensional rotations and half-rotation with the above group multiplication formulas (3).

Multiplication formulas (3) quaternions U, V determines their name "non-Hamiltonian quaternions of the half-rotation".

The following equalities follow from the above formulas:

$$U \otimes \tilde{U} = \tilde{U} \otimes U = 0;$$
$$V \otimes \tilde{V} = \tilde{V} \otimes V = 2E_4, \quad (4)$$

where о=о+$\bar{о}$ are zero quaternion; $\bar{о}$ is a zero vector.

These equalities show that unit elements in groups of quaternions of U, V are respectively the zero quaternion and the doubled single quaternion $2E_4$, and reverse quaternions U^{-1}, V^{-1} are equal to the conjugate $\tilde{U}$, $\tilde{V}$.

Non-Hamiltonian quaternion algebra of the half-rotation

Unnormalized quaternions space U, V, together with their multiplication formulas (3) (the algebraic operations), determined the actual new, associative, non-commutative and unnormalized *group* [11, p. 259] *of quaternions algebras of half-rotation* with single-valued division and without zero divisors [11; 12] (since these group algebras and group there is no zero divisors).

Multiplicity of the quaternions U, V forms a linear four-dimensional Euclidean vector space, while the Hamiltonian quaternions rotation Λ not form a vector space, since haven't zero quaternions.

By analogy with the algebra of Hamiltonian quaternions Λ of rotation the exceptional quaternions algebras U, V of half-rotation are further endowed [4, p. 103–104] the structures of:

1) the commutative group under addition;

2) the non-commutative, associative four-dimensional algebra of division over the real. Thus the operations of addition and multiplication group (3) are distributive [3, p. 32].

Application of non-Hamiltonian quaternions of the half-rotation in the problems of the control orientation

Parameters of quaternions of U, V are used for the solution of tasks of control of orientation of the spacecraft (SC), as solid body, in positive definite quaternion functions f_u and f_v Lyapunov of a square look [5; 10]:

$$f_u = \alpha_u u_0^{\ 2} + \beta_u \left(\bar{\lambda} \cdot A_u \bar{\lambda}\right) + \gamma_u \left(\bar{\omega} \cdot \bar{g}\right);$$

$$f_v = \alpha_v v_0^{\ 2} + \beta_v \left(\bar{\lambda} \cdot A_v \bar{\lambda}\right) + \gamma_v \left(\bar{\omega} \cdot \bar{g}\right), \quad (5)$$

where α_u, β_u, $\gamma_u > 0$ and α_v, β_v, $\gamma_v > 0$; A_u, A_v are definitely positive symmetric constant operators; is the momentum kinematics vector of the spacecraft; J is the operator (tensor) of inertia of the spacecraft; $\bar{\omega}$ is the angular velocity vector of the spacecraft.

To ensure control shortest reversals spacecraft function is used f_u when $u_0 < 1$, $v_0 > 1$ $(0 < \varphi < \pi)$, or function f_v when $u_0 > 0$, $v_0 < 0$ $(\pi < \varphi < 2\varphi)$.

With an appropriate choice of formulas determine the vector of control points (as described, for example, in [10]) a negative definition of the derivative of Lyapunov functions in time provides. The result is the asymptotic stability of the processes controlling the orientation of the spacecraft and its shortest spreads throughout the range of variation of the angle from 0° to 360°.

Application of non-Hamiltonian quaternions of the half-rotation in the algorithms of the orientation determine

Parameters – the coordinates of exceptional quaternions U, V used in control algorithms by orientation of spacecraft are calculated on computer algorithms of SIOS with are similar known algorithm for computing the classical quaternions rotation Euler (Rodrigues – Hamilton) parameters [1; 2; 7–9; 14–18]. This calculation algorithms parameters quaternions U, V easily obtained from the many known algorithms for calculating parameters of Euler (Rodrigues – Hamilton) by simply replacing the scalar parameter λ_0 on the parameters u_0 and v_0, respectively.

Based quaternion U, V may also be prepared by new biquaternions SINS algorithms [1].

The one-step algorithms of the third and fourth orders of accuracy in the “scaled” [9, p. 78, 79] quaternion type $0{,}5U$ used in the “HARTRON” Corp. (Kharkov, Ukraine), in the task of determining the orientation of the spacecraft [13].

Of particular practical interest now becomes a four-step algorithm of the fourth – sixth order accuracy [1; 2; 9; 14; 15; 17; 18], it is possible recurrence computing quaternion U, V with a time step $H = 4h$ (h – a constant and minimum possible sample rate in the computer SINS of signals gyroscopes in time).

The article [7; 8] shows that the four-step algorithms are more effective for use in SINS than the one-step, two-step and three-step algorithms. These algorithms are used intermediate orientation parameters [9, p. 144] – the coordinates $\varphi_{N+4,k}$ ($k = 1, 2, 3$) small vector $\bar{\varphi}_{N+4}$ characterizing finite Euler rotation of the object to a small angle for a time equal to step H. The algorithms for computing these parameters may be represented by a generalized four-step algorithm of the form [9, p. 172]

$$\varphi_{N+4} = q_{N+4} + a_1 Q_{-1} q_1 + a_2 Q_{-2} q_2 + a_3 \left(Q_{-2} q_1 + Q_{-1} q_2\right) + a_4 \left(Q_{-2} q_{-1} + Q_1 q_2\right), \quad (6)$$

where $q_{N+4} = q_{-2} + q_{-1} + q_1 + q_2$; q_{-2}, q_{-1}, q_1, q_2 are column matrix, composed of angular increments corresponding quasi-coordinates (gyro signal) q_α (α = –2, –1, 1, 2) generated in the on-board computer SIOS or SINS on four successive "small" steps h poll gyroscopes; Q_{-2}, Q_{-1}, Q_1, Q_2 are the corresponding skew-symmetric matrix.

The values of the constant coefficients a_v (v = 1…4) to (6), that determine the specific form of the case considered algorithms fourth order of accuracy [9, p. 173], are presented in Table 1 in the form of fractions. Algorithms 1, 2, 3, 5 are given in [9, p. 169; 153; 173; 157], the algorithm 4 – article [14] ("smoothing" algorithm of the fourth order obtained on the basis of Chebyshev polynomials).

Algorithm 3 was first published in 1986 [7] and was also considered in the paper [8] (1987). The monograph [9, p. 158] in the algorithm (3) under the number (3.3.45) contains a typo (instead of the coefficient a_4 = 32/45 printed a_4 = 32/55).

Table 2 shows for comparison the values of constant speed calculation drift of the algorithms (with conical vibrations of SIOS gyroscopes block [14] with conditions: nutation angle – 1 deg, the frequency of vibrations of tapered – 10 Hz, step with computing – 0,01 s) obtained in computer simulations by the method of the parallel accounts [9, p. 218]. As can be seen from Table 2, algorithm 3 is significantly superior in accuracy and other algorithms are substantially so-called *conical algorithm* [15] (the actual sixth-order of accuracy). Further analysis showed the benefits of the algorithm 3 and also in operation performance [8; 9].

The algorithm 3 (as the main part of the calculation algorithm parameters Rodrigues-Hamilton) has been implemented [2, p. 316] in the laser system "SINS-85" in serial production [16; 19; 20] since 2002 and is designed for use on aircraft Il-96-300, Tu-204, Tu-334. Modification of "SINS-85" ("SINS-77", "SIMS-T", "SINS SP-1", "SINS SP-2") are used on the aircraft An-70, Tu-95, Tu-160, Tu-214, Su-35, T-50, Yak-130 [21].

Of particular interest is the possibility of using adaptive conical algorithms [18] for the calculation of the parameters non-Hamiltonian quaternions of half-rotation in SINS. There is the only one optimal among the four-step algorithms the best in terms of accuracy and operation performance adaptive algorithm conical (algorithm 6 of the 6th order from Tables 1, 2). It is obtained based on the algorithm (6) with coefficients insist on a conical motion. This configuration by choosing values of the coefficient b_{23} in the formulas (3.3.107) of [9, p. 173]. This algorithm is performed complete (ideal) compensation conical error due coefficients k_{05}, k_{14}, k_{23} in square terms of the asymptotic estimates (4.3.31) constant speed computing drift-order terms $O(h^6)$ when $\vartheta \to 0$ (ϑ – nutation angle) [9, p. 215]. The accuracy of the algorithm, as shown by computer simulation exceeds the accuracy of the algorithm 3 a decimal ($2{,}2\cdot10^{-5}$ deg/h) under the conditions of calculation, the relevant Table 2.

Table 1

The constant coefficients of four-step algorithms

Factors	Number of algorithm					
	1	2	3	4	5	6
a_1	0	0	22/45	184/315	–74/45	534/945
a_2	16/9	0	22/45	112/315	–9/2	486/945
a_3	0	4/3	22/45	212/315	86/45	414/945
a_4	0	0	32/45	52/105	0	696/945

Table 2

The constant velocity of the drift computing of four-step algorithms

Option	Number of algorithm					
	1	2	3	4	5	6
The actual order of accuracy	4	4	6	6	6	6
The drift velocity, deg/h	2,5	1,4	$3{,}9\cdot10^{-4}$	$9{,}6\cdot10^{-2}$	$1{,}1\cdot10^{-2}$	$2{,}2\cdot10^{-5}$

Optimum conical algorithm 6 exceeds the accuracy even of the four-step algorithm American company Litton [15] providing for filtering signals of laser gyroscopes [21; 22]. The computational complexity of optimal algorithm 6 equal to the computational complexity of the algorithm 4 and the algorithm of the company Litton.

There is also the only one among the five-step algorithms the optimal conical algorithm of 6th order with the ideal correction of the conical error. A method for constructing such an algorithm and computer study of its accuracy and operation performance based on asymptotic estimates of similar cases four-step algorithm [9, p. 218, p. 249–255].

Conclusion

The possibility of using non-Hamiltonian quaternions of half-rotation in strapdown inertial guidance and control is shown. In contrast to the classical Hamiltonian normalized quaternions of rotations the considered non-Hamiltonian half-rotation quaternions can be zero and their modules and norms depend on the corner of the end Euler rotation.

The parameters of the non-Hamiltonian quaternions of half-rotation are appropriate to use in advanced SIOS and SINS of aerospace aircrafts, along with the classic parameters of Euler (Rodrigues–Hamilton), or instead of them.

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