STUDY OF NAVIER – STOKES EQUATION SOLUTION III. THE PHYSICAL SENSE OF THE COMPLEX VELOCITY AND CONCLUSIONS

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In this part of the article is described the physical meaning of complex velocity. The relationship between the Schrödinger equation and the Navier – Stokes equation obtain. Schrödinger equation and the Navier – Stokes equations are, in general, have a countable number of turbulent energy and solutions.

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Physical Meaning of Complex Solution

Let us explain physical meaning of the complex turbulent solution. So, we will consider real solution of ordinary differential equations system $x_\alpha(t)$.

Let us assume that initial data have an average value x_α^0 and mean root square $\left\langle \left[\Delta x_\alpha^0\right]^2 \right\rangle$.

Mean root square of initial data for Navier – Stokes equation is defined by surface roughness or by initial data which are not precisely defined. Then, for mean root square of the solution we have

$$\left\langle [\Delta x_l]^2 \right\rangle = \left\langle [x_l - \langle x_l \rangle]^2 \right\rangle =$$

$$= \left\langle x_l^2 \right\rangle - 2\langle x_l \rangle \langle x_l \rangle + \langle x_l \rangle^2 = \left\langle x_l^2 \right\rangle - \langle x_l \rangle^2 .$$

Then

$$\left\langle x_l^2 \right\rangle = \langle x_l \rangle^2 + \left\langle [\Delta x_l]^2 \right\rangle = \left| \langle x_l \rangle + i\sqrt{\left\langle [\Delta x_l]^2 \right\rangle} \right|^2 . \quad (1)$$

Here I will provide the formulation of inverse Pythagoras theorem. For any three of positive numbers a, b and c, such that $a^2 + b^2 = c^2$, there is a rectangular triangle with legs a and b and hypotenuse c. Hence, mathematical mean value and mean square deviation form legs and hypotenuse is an average square root of the value. That is, average $\langle x_l \rangle$ is orthogonal to mean square deviation $\sqrt{\left\langle [\Delta x_l]^2 \right\rangle}$ which forms imaginary part of the body coordinate. Thus, the Cartesian space with oscillatory high frequency velocity (period of fluctuation is less than measurement time), obtained as a result of averaging in time, becomes complex space. That is, in case of large mean root square of the real space, it should be considered as complex three-dimensional space where imaginary part corresponds to mean square deviation. At the same time, there is following relation between variables $\sqrt{\left\langle x_l^2 \right\rangle} = \left(\langle x_l \rangle + i\sqrt{\left\langle [\Delta x_l]^2 \right\rangle}\right)\alpha$; $|\alpha| = 1$,, and the complex number α is chosen in such a way that the imaginary part had positive or negative value. Mean square deviation satisfies this condition. But sometimes the mean square deviation is positive, for example, in case of dielectric permeability where positive and negative charges have an influence. In this case we have a formula $\varepsilon = \varepsilon_0 + \dfrac{4\pi i\sigma}{\omega}$ where real part is proportional to positive mean square dipole deviation and conductivity is proportional to average value. But conductivity is divided by frequency which has positive and negative sign.

Therefore, algorithm for finding of average solution or average solution in phase space and its mean root square is reduced to finding of complex solution. The average solution corresponds to real part of solution, and second power of complex part corresponds to mean root square of the solution. This is physical meaning of complex solution, real part is an average solution, and imaginary part is a mean square deviation. And real and imaginary parts are orthogonal and form complex space. Really, according to inverse Pythagoras theorem, due to formula (1) mathematical mean value and mean square deviation form legs and average square is a hypotenuse.

Here we would like to note that when calculating the flow motion and one term of a series is taken into account, it is necessary to take square root of imaginary part as forward velocity is calculated. The imaginary part corresponds to square root of oscillatory part of dimensionless velocity.

This situation is similar to calculation of deviation at random choice of forward or back step with probability 1/2 and the point position after N steps is defined by $\sqrt{N}$.

Real and imaginary parts of the solution are located on different axes of complex space. But if you average imaginary dimensionless part, you will have

$$\langle x_\alpha(t)\rangle + i\sqrt{\langle[\Delta x_\alpha(t)]^2\rangle} \to \langle x_\alpha(t)\rangle + i\sqrt[4]{\langle[\Delta x_\alpha(t)]^2\rangle}.$$

And the solution is equal to the module of the last value and, for different roughness, the imaginary part of the solution should be multiplied by averaging multiplier. At the same time, if all coefficients of a series in non-linear equations system are calculated, it is not necessary to calculate square root of imaginary part. It is necessary to summarize complex values and to calculate module of the sum.

Now we will show that the imaginary part of complex derivative of coordinate in phase space of the differential equation forms pulsing coordinate motion in phase space, i.e. in space of variables

$$\langle x_k(t)\rangle + i\sqrt{\langle[\Delta x_k(t)]^2\rangle}.$$

Average values are used for variables as, at molecular level, the medium is not smooth.

Lemma. Complex solution yields fluctuating pulsing function of flow motion coordinates.

The imaginary part of velocity corresponds to rotation speed in phase space. As rotation radius is known, it is also possible to determine rotation frequency. In the rotation plane, complex velocity with constant rotation radius and constant frequency can be written in the form

$$V_x + iV_y = V_0 \exp(i\omega t).$$

In case of varying over the space stationary speed, locally, this formula can be written for one plane as

$$V_x(x,y) + iV_y(x,y) = V_0(x,y)\exp\left[i\int_0^t \omega(x,y,u)du\right],$$

and frequency is dependent on time as the phase shift is provided as a result of harmonic oscillations in neighboring points. Sum of harmonic oscillations with different time-dependent frequencies defines pulsing mode in phase space at stationary complex velocity. That is, this complex velocity defines the coordinates of phase space points pulsing in time. The situation is similar to existence of several stationary vortexes defining the pulsing rotation of the flow.

Lemma 6. Three-dimensional flow velocity can be written in the form

$$V_l = V_{tl} + iV_{nl} = V_l \exp(i\varphi_l);\quad \varphi_l = \arg(V_{tl} + iV_{nl}).$$

And velocity is defined in the form of integral of tangent acceleration by formula

$$V_{tl} = \int_{t_0}^{t} t_l(u) w_t(u) du + V_{tl}(t_0) = \int_{t_0}^{t} t_l(u) \frac{d\sqrt{\sum_{k=1}^{3} V_k(u)V_k^*(u)}}{du} du + V_{tl}(t_0) =$$

$$= \int_{t_0}^{t} t_l(u) \frac{d\sqrt{\sum_{k=1}^{3}\left[V_{tk}^2(u) + V_{nk}^2(u)\right]}}{du} du + V_{tl}(t_0).$$

Integral of perpendicular component of acceleration defines perpendicular component of velocity by formula

$$V_{nl} = \operatorname{Im} V_l(\tau_0) = \int_{\tau_0}^{\tau} w_{nl}(u)du = \int_{\tau_0}^{\tau} \frac{n_l(u)|\operatorname{Im}\mathbf{V}|^2}{\rho(u)} du = \int_{s_0}^{s} |\operatorname{Im}\mathbf{V}(s)| \frac{n_l(s)}{\rho(s)} ds =$$

$$= \int_{s_0}^{s} |\operatorname{Im}\mathbf{V}| dt_l = \begin{cases} |\operatorname{Im}\mathbf{V}|[t_l(s) - t_l(s_0)], & |\operatorname{Im}\mathbf{V}| = \text{const} \\ \int_{s_0}^{s} |\operatorname{Im}\mathbf{V}| dt_l, & |\operatorname{Im}\mathbf{V}| \neq \text{const} \end{cases};$$

$$\sum_{k=1}^{3}\left[V_{tk}^2(u) + V_{nk}^2(u)\right] = |\mathbf{V}|^2;$$

$$dt_l(s) = \frac{n_l(s)ds}{\rho(s)};$$

$$t_l(\tau) = \frac{\operatorname{Im}\mathbf{V}_l}{|\operatorname{Im}\mathbf{V}|}.$$

At that value of local velocity is $V_{nl}(\tau_0) = \mathrm{Im}V_l(\tau_0)$, $V_{tl}(\tau_0) = \mathrm{Re}(V_l)\tau_0$. But value of velocity obtained as a result of integration of centripetal acceleration is not zero $\left(V_{nl}(\tau) \neq 0\right)$, but this velocity become equal to zero for the same initial point, at constant particle velocity and constant curvature radius with rotation period $T = \dfrac{2\pi R}{|\mathbf{V}|}$, where R – curvature radius. For variable particle velocity depending on time, when one of the integrals $\int_{\tau_0}^{\tau} |\mathbf{V}|\, dt_l = 0$, which, at finite curvature radius of one sign of the trajectory, is finite and equal to

$$T = \int_0^{2\pi} \frac{R(\varphi)\, d\varphi}{|\mathbf{V}(\varphi)|} = \int_{s_0}^{s_0+s_T} \frac{ds}{|\mathbf{V}(s)|}; \quad 2\pi = \int_{s_0}^{s_0+s_T} \frac{ds}{R(s)},$$

as tangential direction t_l, changes sign in the course of rotation.

Tangential acceleration is defined by formula

$$w_t = d\sqrt{\sum_{k=1}^{3}\left[V_{tk}^2(t) + V_{nk}^2(t)\right]} \Big/ dt.$$

Direction of velocities ΔV_{tl}, ΔV_{nl} is orthogonal, their sum yields increase of motion velocity module

$$\sum_{l=1}^{3}(dV_l)^2 = \sum_{l=1}^{3}\left[(dV_{tl})^2 + (dV_{nl})^2\right] = \sum_{l=1}^{3}\left|dV_{tl} + i dV_{nl}\right|^2,$$

as $$\sum_{l=1}^{3}(w_l)^2 = \sum_{l=1}^{3}\left[(w_{tl})^2 + (w_{nl})^2\right].$$

Components of these projections, differentiable with respect to time, define tangential and orthogonal accelerations. At the same time, concepts of tangential and orthogonal velocities are entered which, in the Cartesian space, are not orthogonal to $(\mathbf{V}_t, \mathbf{V}_l) \neq 0$, but in six-measured complex space they are orthogonal, and their module of complex vector $V_l = V_{tl} + iV_{nl}$ is equal to

$$\sum_{l=1}^{3}|V_l|^2 = \sum_{l=1}^{3}\left[(V_{tl})^2 + (V_{nl})^2\right] = \sum_{l=1}^{3}\left|V_{tl} + iV_{nl}\right|^2.$$

It can be proved by use of expression $\mathbf{V}_t = \sum_{l=1}^{3} V_{tl}\mathbf{e}_{tl}$, $\mathbf{V}_n = \sum_{l=1}^{3} V_{nl}\mathbf{e}_{nl}$ and calculation of module as product of complex conjugate vectors taking into account orthogonality of six real unit vectors.

Conclusions

Thus, solution of Navier – Stokes equations for not multiple balance positions is obtained. It is defined by expressions

$$\mathbf{V}(t,\mathbf{r}) = \sum_{n=1}^{N} \mathbf{x}_n(t)\varphi_n(\mathbf{r});$$

$$\sum_{s=1}^{S} \lambda_l^s \ln\left(x_l - a_l^s\right)\Big|_{t_0}^{t} = H_l(t,t_0), \quad l = 1, \ldots, 2N;$$

$$\lambda_l^s = \frac{1}{\left(a_l^s - a_l^1\right)\ldots\left(a_l^s - a_l^{s-1}\right)\left(a_l^s - a_l^{s+1}\right)\ldots\left(a_l^s - a_l^S\right)},$$

where values a_l^s are coordinates of balance positions.

Laminar solution corresponds to the solution of linear problem with convective term averaging; structure of turbulent solution is

$$\mathbf{V}(t,\mathbf{r}) = \sum_{n=1}^{N}\sum_{k=-\infty}^{\infty} \frac{a_{nk}}{g(t) - g_{nk}(t_n)}\varphi_n(\mathbf{r}) + \mathbf{a}^s,$$

where $g_{nk}(t)$ – known defined continuous function, value of $g_{nk}(t_n) = g_k\left(t_0, x_k^0\right) + \pi n$ is defined from initial conditions, and $\lim_{t\to\infty} g(t) = \infty$. At that, the solution contains a lot of poles which, for real solution and real initial data, yield infinity.

At real time and complex initial conditions which define complex value of $g_{nk}\left(t_0, x_k^0\right)$, and, as $g(t)$ is real, the complex solution is finite.

At that, formula

$$\sum_{s=1}^{S} \lambda_l^s \ln\left(x_l - a_l^s\right)\Big|_{t_0}^{t} = H_l(t,t_0), \quad l = 1, \ldots, N. \quad (2)$$

may have branching points in which the solution continuously passes into other branch of the solution. This does not contradict the theorem of solution uniqueness for Cauchy problem as the left part of the differential equation tends to infinity in branching point. Derivative of right part of ordinary differential equation also tends to infinity in branching point. So we have a point of discontinuous solution. But this solution can be continued by a formula (2).

This situation is similar to Schrödinger equation when generally we have finite number of solutions. It is not surprising as Schrödinger equation can be reduced to Navier – Stokes equation. Now we will prove it. For this we will write down Schrödinger equation and will transform it using equality

$$\frac{\partial^2 \psi}{\partial x_l^2} = \psi\left[\frac{\partial^2 \ln\psi}{\partial x_l^2} + \frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x_l}\right)^2\right];$$

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\sum_{l=1}^{3}\frac{\partial^2\psi}{\partial x_l^2}+U\psi=-\frac{\hbar^2}{2m}\psi\sum_{l=1}^{3}\left[\frac{\partial^2\ln\psi}{\partial x_l^2}+\frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x_l}\right)^2\right]+U\psi.$$

Dividing the equation by mass $m\psi$ we obtain the equation

$$i\frac{\hbar}{m}\frac{\partial\ln\psi}{\partial t}+\frac{\hbar^2}{2m^2}\sum_{l=1}^{3}\left(\frac{\partial\ln\psi}{\partial x_l}\right)^2=-\frac{\hbar^2}{2m^2}\sum_{l=1}^{3}\frac{\partial^2\ln\psi}{\partial x_l^2}+\frac{U}{m}.$$

Now we will write a private derivative equation, will take a gradient of both parts of equation and will enter real velocity to the formula

$$\mathbf{V}=-i\frac{\hbar}{m}\nabla\ln\psi;$$

$$\frac{\partial i\frac{\hbar}{m}\nabla\ln\psi}{\partial t}+\frac{\hbar^2}{m^2}\sum_{l=1}^{3}\frac{\partial\ln\psi}{\partial x_l}\frac{\partial\nabla\ln\psi}{\partial x_l}=\frac{i\hbar}{2m}\sum_{l=1}^{3}\frac{\partial^2 i\frac{\hbar}{m}\nabla\ln\psi}{\partial x_l^2}+\frac{\nabla U}{m}.$$

Substituting velocity value into transformed Schrödinger equation, we have

$$\frac{\partial V_p}{\partial t}+\sum_{l=1}^{3}V_l\frac{\partial V_p}{\partial x_l}=v\sum_{l=1}^{3}\frac{\partial^2 V_p}{\partial x_l^2}-\frac{\partial U}{\partial x^p}\Big/m;\quad v=\frac{i\hbar}{2m}.$$

Now we have three-dimensional Navier – Stokes equation with pressure corresponding to potential. Nevertheless, the hydrodynamic problem differs from the equation of Navier – Stokes derived from Schrödinger equation and continuity equation.

At the same time it is possible to draw an analogy between laminar single-value mode and free, single-value description of bodies.

Between turbulent mode, having finite number of solutions, and description of bound particles having finite number of solutions. In case of turbulent complex and laminar real modes there is a boundary between them and critical Reynolds number. The similar boundary is available between free and bound particles description, which corresponds to energy transition from negative to positive state. In turn, Navier – Stokes equation has to have discrete energy levels of turbulent flow states, transitions between these states with energy emission or absorption have to be realized.

The boundary between free particles description and bound particles description can be defined, this is transition to complex quantum number or to infinity of the main quantum number of hydrogen atom. At that, infinite quantum number of hydrogen atom, passing through zero value of expression $1/n^2$ where n – main quantum number, becomes imaginary and continuous. Wave function of free motion, which is continuous at continuous energy, corresponds to laminar solution of hydrodynamic problem for which single valued solution exists. And for large quantum number, the system is quasi-classical, i.e. for quantum number which is close to boundary (quantum number is equal to infinity) system is almost classical.

And there is a boundary between free solution and solution which describes bound states. This is zero energy value and, likewise nonlinear private derivatives equations, boundary exists between turbulent complex solution and laminar real solution.