STUDY OF NAVIER – STOKES EQUATION SOLUTION I. THE GENERAL SOLUTION OF NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Values of large dimensionless unknown functions (for example, a large Reynolds number) can be found out as solutions of non-linear partial differential equation. In this case these equations can be brought to some number of non-linear ordinary differential equations. Turbulent solutions corresponding to large values of unknown function are complex. Transition from real solution to complex turbulent solution is realized through infinity of the right parts of ordinary differential equation system to which Navier – Stokes equations are brought. Thus, real solution of Navier – Stokes equation for turbulent mode yields function going to infinity. At the same time, complex solution for the turbulent mode is finite. Fluid flow resistance coefficient is calculated for round pipeline with different pipeline walls roughness.

Keyword: large dimensionless unknown functions, solutions of non-linear partial differential equation, Navier – Stokes equation, turbulent function, fluid flow resistance coefficient, round pipeline

Problem Formulation

Let us consider Navier – Stokes problem and continuity equation for incompressible fluid. They are as following

$$\frac{\partial \mathbf{V}_k(t,\mathbf{r})}{\partial t} + \sum_{l=1}^{3} \mathbf{V}_l(t,\mathbf{r}) \frac{\partial \mathbf{V}_k(t,\mathbf{r})}{\partial x_l} = -\frac{\partial P(t,\mathbf{r})}{\rho \partial x_k} + \nu \Delta \mathbf{V}_k(t,\mathbf{r}), \quad k = 1,...,3.$$

$$(\nabla, \mathbf{V}) = 0$$

Boundary conditions on the body boundary adjoining to fluid are $\mathbf{V}(t, \mathbf{r}) = 0$, $\mathbf{r} \in S$ where S is a body boundary. We will seek a solution in the form of series using Galerkin method (hereinafter $N \to \infty$)

$$\mathbf{V}_l(t,\mathbf{r}) = \sum_{n=1}^{N} \mathbf{x}_{nl}(t) \varphi_{nl}(\mathbf{r}), P(t,\mathbf{r}) = \sum_{n=1}^{N} y_n(t) \psi_n(\mathbf{r}) + \psi_0(\mathbf{r});$$

$$\mathbf{r} \in S \to \varphi_n(\mathbf{r}) = 0, \varphi_n(\mathbf{r}), \psi_n(\mathbf{r}) \in C^2,$$

where space C^2 is twice continuously differentiable function, $\psi_0(\mathbf{r})$ is a defined external action which, in case of pipeline, is equal to

$$\psi_0(\mathbf{r}) = \frac{-(P - P_0) z}{L} + P,$$

where z – direction of the pipeline longitudinal; P, P_0 – pressure at the beginning and the end of the pipeline; L – pipeline length.

Now we substitute these functions into the differential equation, multiply by $\psi_m(\mathbf{r})$ and integrate over the volume, then we obtain following differential equations system:

$$\frac{dx_m(t)}{dt} = \sum_{p,q=1}^{3N} F_{mpq} x_p(t) x_q(t) + \sum_{p=1}^{4N} G_{mp} x_p(t) + H_m, \quad m = 1,..., 3N;$$

$$\sum_{p=1}^{3N} P_{mp} x_p(t) = 0, \quad m = 1,..., N. \qquad (A.1)$$

After we resolved the second equation (A.1), substituted

$$x_{n+2N}(t) = \sum_{m=1}^{2N} c_{nm} x_m(t), \quad n = 1,..., N$$

from the second equation (A.1) to the first one, we have

$$\frac{dx_m(t)}{dt} = \sum_{p,1=q}^{2N} F^1_{mpq} x_p(t) x_q(t) + \left(\sum_{p=1}^{2N} + \sum_{p=3N+1}^{4N} \right) G^1_{mp} x_p(t) + H_m, \quad m = 1,..., 2N;$$

$$\sum_{p,1=q}^{2N} F^1_{mpq} x_p(t) x_q(t) + \left(\sum_{p=1}^{2N} + \sum_{p=3N+1}^{4N} \right) G^1_{mp} x_p(t) + H_m = 0, \quad m = 2N+1, ..., 3N. \qquad (A.2)$$

Defining $x_{n+3N}(t)$, $n = 1,..., N$ corresponding to pressure change, from the second equation (A.2) and substituting found out value into the first equation (A.2), we have equations system

$$\frac{dx_m(t)}{dt} = \sum_{p,q=1}^{2N} F^2_{mpq} x_p(t) x_q(t) + \sum_{p=1}^{2N} G^2_{mp} x_p(t) + H^1_m, \quad m = 1,..., 2N. \qquad (A.3)$$

At that

$$x_{n+2N}(t) = \sum_{m=1}^{2N} c_{nm} x_m(t); \quad x_{n+3N}(t) = \sum_{m=1}^{N} c_{nm} H_m + \sum_{m=1}^{2N} b_{nm} x_m(t) + \sum_{p,q=1}^{2N} F^1_{mpq} x_p(t) x_q(t), \quad n = 1,..., N.$$

Where values

$$\mathbf{x}_{nl}(t) = x_{n+N(l-1)}(t), \quad l = 1,..., 3; \qquad y_n(t) = x_{n+3N}(t), \quad n = 1,..., N.$$

are known and coefficients F_{mpq}, G_{mp}, H_m, F^1_{mpq}, G^1_{mp}, H^1_m, F^2_{mpq}, G^2_{mp}, H^2_m, c_{nm}, b_{nm} are constants. This system of non-linear ordinary differential autonomous equations (A.3) is to be solved. Solution convergence issues will be discussed below in the text.

Finding of Solution of Ordinary Differential Equations in Complex Plane

Let us consider system of non-linear differential autonomous equations

$$\frac{dc_l}{dt} = Q_l(c_1,..., c_N), \; l = 1, ..., N. \qquad (1)$$

Navier – Stokes equation system and continuity equation can be brought to system of non-linear differential equations:

$$\frac{dc_m(t)}{dt} = \sum_{p,q=1}^{N} F_{mpq} c_p(t) c_q(t) + \sum_{p=1}^{N} G_{mp} c_p(t) + H_m = Q(c_1,..., c_N), \quad m = 1,..., N, \qquad (2)$$

where three-dimensional velocity is defined by formula $\mathbf{V}(t, x_1, x_2, x_3) = \sum_{n=1}^{N} \mathbf{c}_n(t) \varphi_n(x_1, x_2, x_3)$.

At that, function $\varphi_n(x_1, x_2, x_3)$ is given in the form of sine or cosine. Then coefficients $c_n(t)$ for continuous function decrease not more rapidly than $1/n^2$ when index increases and series reduction is possible, i.e. instead of infinite number of terms, finite terms number is used. At the same time, the infinite number of terms forms convergent series.

It was found out that a set of $N+1$ coordinates for the system balance position exists (2). Indeed, let us assume that we have found several balance positions with coordinates b^0_l, $l = 1, ..., N$. Let us seek the solution in the form $b_l = b^0_l + b^s_l$. For that we will substitute the solution into the right part of the differential equation (2) and will equate it to zero, then following equations system is obtained

$$\sum_{l=1}^{N} A_{kl}\left(b^s_1,..., b^s_N\right) b^s_l = 0.$$

For existence of non-zero solution of this differential equation, it is necessary that determinant is equal to zero:

$$\left| A_{kl}\left(b^s_1,..., b^s_N\right) \right| = 0.$$

Given zero determinant, coefficients b^s_l from linear equation will be defined up to a multiplier. This multiplier will be defined from equality to zero of determinant of non-linear equation system. Thus, we have N unknown multipliers, which will be defined from determinant equality to zero. I.e. set of $N+1$ coordinates of the system balance position exists.

Differential equation system (2) for non-multiple balance positions can be expressed by means of $c_l = \sum_{k=1}^{N} g_{lk} x_k$ substitution. At that, the system (2) balance positions b^s_l, $l = 1, ..., N$, $s = 1, ..., S$ will be transformed into balance positions a^s_l, $l = 1, ..., N$, $s = 1, ..., S$ and eigen values and eigen vectors of the linearized system (2) will be defined as.

$$\left[\frac{\partial Q_k}{\partial c_m}\left(b^s_1,..., b^s_N\right) - \Lambda^s_\alpha \delta_{km} \right] g^s_{m\alpha} = 0;$$

$$\left| \frac{\partial Q_k}{\partial c_m}(b^s_1,..., b^s_N) - \Lambda^s_\alpha \delta_{km} \right| = 0.$$

Equation system (2) will be

$$\frac{dx_n}{dt} = \Lambda_n^s\left(x_n - a_n^s\right) + \sum_{k=1}^{N}\left(x_n - a_n^k\right)^2 P_n^k(x_1,\ldots,x_N) = F_n(x_1,\ldots,x_N). \tag{3a}$$

Values a_l^s satisfy condition $F_k\left(a_1^s,\ldots,a_N^s\right) = 0,$. $k = 1, \ldots, N$, $s = 1, \ldots, S$.

Equation system (3a) can be written as

$$\frac{dx_l}{dt} = \exp\left[G_l(x_1,\ldots,x_N)\right]\prod_{s=1}^{S}\left(x_l - a_l^s\right), \tag{3b}$$

where multiplier which can never be equal to 0 is used – $\exp\left[G_l(x_1,\ldots,x_N)\right]$, and this multiplier is equal to

$$\exp\left[G_l(x_1,\ldots,x_N)\right] = \frac{F_l\left(x_1,\ldots,x_N\right)}{\prod_{s=1}^{S}\left(x_l - a_l^s\right)}.$$

After this multiplier is substituted to (3b) we obtain (3a). Now we will demonstrate that this multiplier can never be equal to 0. When $x_l \to a_l^\alpha$, $l = 1, \ldots, N$ the limit of

$$\exp\left[G_l\left(a_1^\alpha,\ldots,a_N^\alpha\right)\right] = \frac{\partial F_l\left(a_1^\alpha,\ldots,a_N^\alpha\right)}{\partial x_l}\Big/\left[\left(a_l^\alpha - a_l^1\right)\ldots\left(a_l^\alpha - a_l^{\alpha-1}\right)\left(a_l^\alpha - a_l^{\alpha+1}\right)\ldots\left(a_l^\alpha - a_l^S\right)\right] =$$

$$= \frac{\Lambda_l^\alpha}{\left(a_l^\alpha - a_l^1\right)\ldots\left(a_l^\alpha - a_l^{\alpha-1}\right)\left(a_l^\alpha - a_l^{\alpha+1}\right)\ldots\left(a_l^\alpha - a_l^S\right)}$$

is finite.

Here we canceled out a multiplier $x_l - a_l^\alpha$, as we consider only not coincident roots being coordinates of balance position. So we showed that this multiplier does not equal to zero after infinite time.

Thus, the differential equation can be written as

$$\frac{dx_l}{dH_l(t,t_0)} = \prod_{s=1}^{S}\left(x_l - a_l^s\right); \quad \frac{dH_l(t,t_0)}{dt} = \exp\left\{G_l\left[x_1(H_l),\ldots,x_N(H_l)\right]\right\}, \quad l = 1, \ldots, N, \tag{4}$$

where $H_l(t, t_0)$ – function which tends to infinity when coordinates tend to balance position. For real solutions, this function is monotonic. That is, we have obtained dependence of the solution on value $H_l(t, t_0)$, which is monotonic time-dependent function.

Lemma 1. Necessary and sufficient criterion for unknown function to tend to steady balance position coordinates is $H_l(t, t_0) \to \infty$ when $t \to \infty$. At the same time, balance position coordinates have to have a real part.

So, we have

$$\exp\left[G_l\left(x_1,\ldots,x_N\right)\right] \to \exp\left[G_l\left(a_1^s,\ldots,a_N^s\right)\right] =$$

$$= \frac{\Lambda_l^s}{\left(a_l^\alpha - a_l^1\right)\ldots\left(a_l^\alpha - a_l^{\alpha-1}\right)\left(a_l^\alpha - a_l^{\alpha+1}\right)\ldots\left(a_l^\alpha - a_l^S\right)}; \tag{5}$$

at $t \to \infty$ and hence $H_l(t, t_0) \to \infty$, $l = 1, \ldots, N$ as integral of constant. Inverse theorem is also valid, on condition $H_l(t, t_0) \to \infty$, $l = 1, \ldots, N$, one of steady balance positions is realized. This is a consequence of solution type; on condition $H_l(t, t_0) \to \infty$, $l = 1, \ldots, N$, according to Lemma 3, negative real part of value λ_l^s exists in formula (6) and solution tends to balance position coordinate a_l^s in formula (4). If balance position coordinates have real parts, values λ_l^s have real part. At that $t \to \infty$.

Lemma 2. Solution of differential equation (1) is function $x_l(t)$ which satisfies to equation (6).

To obtain (6), let us divide equation (4) by product of multipliers $x_l - a_l^s$ and multiply (4) by $dH_l(t, t_0)$. Then we will decompose obtained fraction into sum of simple fractions and perform integration. The following equation is obtained

$$\sum_{s=1}^{S}\lambda_l^s\left[\ln\left(x_l - a_l^s\right) + 2\pi i n_s\right]\Big|_{t_0}^{t} = H_l(t,t_0),$$

$$l = 1, \ldots, 2N.$$

Here for the case of sound energy emission in interval $[t_0, t]$ different branches of logarithm are obtained.

After the expression exponentiated, we have (6)

$$\frac{\prod_{s=1}^{S}\left(x_l - a_l^s\right)^{\lambda_l^s}\exp(2\pi i\lambda_l^s\Delta n_s)}{\prod_{s=1}^{S}\left(x_l^0 - a_l^s\right)^{\lambda_l^s}} = \exp\left[H_l(t, t_0)\right];$$

$$\lambda_l^s = \frac{1}{\left(a_l^s - a_l^1\right)\ldots\left(a_l^s - a_l^{s-1}\right)\left(a_l^s - a_l^{s+1}\right)\ldots\left(a_l^s - a_l^S\right)}, \qquad (6)$$

where all values of balance position coordinates are not multiple and are not dependent on radiation process occurring in an interval $[t_0, t]$. In case of laminar real solution, radiation will not appear, and in case of turbulent solution, followed by radiation, there will be energy transition. Really, presence of radiation yields the complex solution which describes turbulent pulsing mode. At that, at solution transformation, turbulent mode is followed by sound noise. Exponential multiplier does not affect balance position coordinates which define stationary solution. Existence of multiplier $\exp\left(2\pi i\lambda_l^s\Delta n_s\right)$ changes calculated main branch of solution for coordinate x_l, but will not affect balance position coordinate.

Lemma 3. Sum of coefficients λ_l^s by index s is equal to zero, i.e. $\sum_{s=1}^{S}\lambda_l^s = 0$.

In case if following fraction decomposed.

$$P(y) = \frac{Q_{S-1}(y)}{\left(y - a_l^1\right)\ldots\left(y - a_l^{s-1}\right)\left(y - a_l^{s+1}\right)\ldots\left(y - a_l^S\right)}.$$

where $Q_{S-1}(y)$ is $S-1$-ordered polynomial. Equation $\sum_{s=1}^{S}\lambda_l^s = 0$ will remain satisfied,

$$\lambda_l^s = \frac{Q_{s-1}\left(a_l^s\right)}{\left(a_l^s - a_l^1\right)\ldots\left(a_l^s - a_l^{s-1}\right)\left(a_l^s - a_l^{s+1}\right)\ldots\left(a_l^s - a_l^S\right)}.$$

Let us prove this. For this let us consider a sum

$$P(y) = \sum_{s=1}^{S}\frac{Q_{S-1}\left(a_l^s\right)\left(y - a_l^1\right)\ldots\left(y - a_l^{s-1}\right)\left(y - a_l^{s+1}\right)\ldots\left(y - a_l^S\right)}{\left(a_l^s - a_l^1\right)\ldots\left(a_l^s - a_l^{s-1}\right)\left(a_l^s - a_l^{s+1}\right)\ldots\left(a_l^s - a_l^S\right)}.$$

This sum is equal to $P(y) = Q_{S-1}(y)$. We write formula for polynomial equal to $Q_{S-1}(y)$, dividing the equation by product $\left(y - a_l^1\right)\ldots\left(y - a_l^S\right)$ we obtain

$$\sum_{s=1}^{S}\frac{Q_{S-1}\left(a_l^s\right)}{\left(a_l^s - a_l^1\right)\ldots\left(a_l^s - a_l^{s-1}\right)\left(a_l^s - a_l^{s+1}\right)\ldots\left(a_l^s - a_l^S\right)\left(a_l^s - y\right)} +$$

$$+\frac{Q_{S-1}(y)}{\left(y - a_l^1\right)\ldots\left(y - a_l^{s-1}\right)\left(y - a_l^s\right)\left(y - a_l^{s+1}\right)\ldots\left(y - a_l^S\right)} = 0.$$

If suppose that $y = a_l^{S+1}$, equality $\sum_{s=1}^{S+1}\lambda_l^s = 0$ is satisfied when $s + 1$ balance position exists.

But to realize the solution, it is necessary to know balance positions of this non-linear equations system. Besides, balance positions can be multiple that changes the solution finding process, it becomes random or chaotic, but we are not going to consider this case. Nevertheless, it is possible to prove the following important theorem.

Theorem 1. Cauchy task is considered under arbitrary real initial conditions for system of orthogonal non-linear ordinary differential equations (1). If system (1) has complex conjugate balance positions with real parts then, for finite real argument t, Cauchy problem solution for the system (1), for real initial conditions, tends to infinity. Then this solution becomes a complex one, tending to balance position in case when complex balance position coordinates have real part. Here the right part of (1) is considered as being a regular function, real for real arguments. This function has finite number of non-multiple balance positions.

Proving

If the system (2) is resolved at non-multiple balance positions then, according to Lemma 2, we have

$$\left\{-2\lambda_{iml}^{s}\arctan\left[\frac{\left(x_l-a_l^s\right)}{b_l^s}\right]+\lambda_{rel}^{s}\ln\left[\left(x_l-a_l^s\right)^2+(b_l^s)^2\right]\right\}\Bigg|_{t_0}^{t}+ \\ +\sum_k \lambda_l^k\ln\left(x_l-c_l^k\right)\Big|_{t_0}^{t}=H_l(t,t_0), \tag{7}$$

where $a_l^s+ib_l^s$ selected complex balance position, c_l^s other balance positions. Coefficients λ_l^s satisfy condition $\sum_s \lambda_l^s=0$, according to Lemma 3. At that, in sum $\sum_{s=1}^{S}\lambda_l^s$ real part value λ_{rel}^s in case of complex solution λ_l^s presents twice as all values λ_l^s satisfy condition $\sum_s \lambda_l^s=0$, so we have formula

$$2\lambda_{rel}^s+\sum_k \lambda_l^k=0.$$

Let us substantiate solution (7). For that we will modify two complex conjugate terms of the solution (for expression simplicity, index l is omitted)

$$\frac{\lambda_{re}^s+i\lambda_{im}^s}{x-a^s-ib^s}+\frac{\lambda_{re}^s-i\lambda_{im}^s}{x-a^s+ib^s}=\frac{2(x-a^s)\lambda_{re}^s-2b^s\lambda_{im}^s}{\left(x-a^s\right)^2+(b^s)^2}. \tag{8}$$

where $\lambda^s=\lambda_{re}^s+i\lambda_{im}^s$. After integration (8) over argument x, we obtain formula (7)

$$\lambda_{re}^s\ln\left[\left(x-a^s\right)^2+(b^s)^2\right]-2\lambda_{im}^s\arctan\frac{x-a^s}{b^s}.$$

The solution is $x_l(t)=a_l^s+b_l^s\tan D_l(t)$, where

$$D_l(t)=\left\{\sum_k \lambda_l^k\ln\left(x_l-c_l^k\right)\Big|_{t_0}^{t}+\lambda_{rel}^{s}\ln\left[\left(x_l-a_l^s\right)^2+(b_l^s)^2\right]\Big|_{t_0}^{t}-H_l(t,t_0)\right\}\Big/2\lambda_{iml}^s=$$

$$=\left\{\sum_k \lambda_l^k+2\lambda_{rel}^s+\sum_k \lambda_l^k\ln\left(1-\frac{c_l^k}{x_l}\right)+\lambda_{rel}^s\ln\left[\left(1-\frac{a_l^s}{x_l}\right)^2+\frac{(b_l^s)^2}{x_l^2}\right]-\right.$$

$$\left.-\sum_k \lambda_l^k\ln\left(x_l^0-c_l^k\right)-\lambda_{rel}^s\ln\left[\left(x_l^0-a_l^s\right)^2+\left(b_l^s\right)^2\right]-H_l(t,t_0)\right\}\Big/2\lambda_{iml}^s;$$

$$\sum_k \lambda_l^k+2\lambda_{rel}^s=0.$$

At that, value of $\sum_k\left(\lambda_l^k c_l^k+2\lambda_{rel}^s a_l^s\right)$ is real due to existence of complex conjugate balance positions. Thus, for $|x_l|\to\infty$ and finite t, we have equation

$$x_l(t)=a_l^s+b_l^s\tan D_l(t). \tag{9}$$

Solution of this equation tends to infinity.

At that, solution of differential equation for rising $H_l(t, t_0)$, according to Lemma 1, can have complex roots

$$\sum_k \lambda_l^k\left[\ln\left(x_l-a_l^k\right)+2\pi i\Delta n_k\right]\Big|_{t_0}^{t}=H_l(t,t_0).$$

At that, as equation $\sum_k \lambda_l^k=0$ is satisfied according Lemma 3 and balance positions have real parts, values with negative real part λ_l^k exist, so convergence to one of the balance positions takes place. Real solution will tend to infinity at that existence, and uniqueness condition for Cauchy problem will be breached. According to Lemma 1, at $H_l(t, t_0)$ infinity, unknown function will tend to one of balance positions. This balance position cannot be real as the real solution is infinite. This means that the solution will have a branching point and will tend to complex balance position. That is, for balance complex positions, finite complex solution is obtained at $H_l(t, t_0)$ change. Thus, in some point a complex solution will begin.

End of the proof.

Now we will give an example describing this property of the differential equation, transition to the complex solution. So, for the differential equation, there can be a complex solution instead of infinite real one

$$\frac{dx}{dt}=1+x^2.$$

And these balance positions are purely imaginary, that is, the solution cannot tend to balance position. And the real solution of this differential equation tends rapidly to infinity

$$x = \tan[t - t_0 + \arctan(x_0)].$$

Using an implicit solution finding scheme, we obtain the following equation

$$x = x_0 + (1 + x^2)\Delta t + 0(\Delta t)^2.$$

Seeking solution in respect to unknown function x, we obtain the following implicit scheme

$$x = \frac{1 - \sqrt{1 - 4\left[x_0 + \Delta t + 0(\Delta t)^2\right]\Delta t}}{2\Delta t}.$$

This implicit scheme with constant step correctly describes solution tendency to infinity. At reduced calculation step, this scheme yields larger value of variable t, that is, it yields larger value of unknown function. That is, it correctly describes behavior of the differential equation solution up to infinity. When infinity is reached, under condition $x_0 > 1/(4\Delta t) - \Delta t - 0(\Delta t)^2$, the finite complex solution will be found. Numerical computation of this equation has validated this analysis of the solution obtained.

At that, the complex solution possesses new properties; it performs complex rotation around balance position. At the same time the real solution tends to infinity, i.e. right part of the differential equation tends to infinity and existence and uniqueness condition for Cauchy problem are breached, so additional complex solution is arisen.

The solution for complex initial data is given by formula

$$x = \tan\left[t - t_0 + \arctan(x_0 + i\delta)\right]$$

for any t. Thus, approximately we have

$$x(t) = -i\frac{\exp\left\{i\left[t - t_0 + \arctan(x_0 + i\delta)\right]\right\} - \exp\left\{-i\left[t - t_0 + \arctan(x_0 + i\delta)\right]\right\}}{\exp\left\{i\left[t - t_0 + \arctan(x_0 + i\delta)\right]\right\} + \exp\left\{-i\left[t - t_0 + \arctan(x_0 + i\delta)\right]\right\}} =$$

$$= i - 2i\exp\left\{2i\left[t - t_0 + \arctan(x_0 + i\delta)\right]\right\} + i\exp\left\{4i\left[t - t_0 + \arctan(x_0 + i\delta)\right]\right\} + \ldots =$$

$$= i - 2i\exp\left[2i(t - t_0 + \alpha) - 2\beta\right] + i\exp\left[4i(t - t_0 + \alpha) - 4\beta\right] + \ldots +$$

$$+ \arctan(x_0 + i\delta) = \alpha + i\beta.$$

If we choose branch with positive β, we obtain converging series. At that, this fraction denominator never becomes zero.

That is, for real plane, finite solution does not exist. In complex plane, finite continuous solution exists in the case if balance positions are not multiple.

But there is a question – what is the physical meaning of imaginary part of the solution?

Physical Meaning of Exact Complex Solution

So, for turbulent solution corresponding to complex balance position coordinates, we have solution

$$x_l = \alpha_l^s + \beta_l^s \tan(D)h_l.$$

The solution consists of step term in the form of delta-function and smooth part

$$x_l = \alpha_l^s + \beta_l^s\left\{\tan\left[D(h_l) - i0\right] - \tan\left[D(h_l) + i0\right]\right\}/2 + \beta_l^s\left\{\tan\left[D(h_l) - i0\right] + \tan\left[D(h_l) + i0\right]\right\}/2.$$

As, at averaging over period, tangents sum without taking into account step term is equal to zero, we will study the step term of the solution. At that, this solution has singularity when condition $D(h_l) = \pi(k + 1/2)$ is satisfied. Step term of the solution is

$$x_l = \alpha_l^s + \beta_l^s\sum_{k=-\infty}^{\infty}\left\{\left[\frac{1}{D(h_l) - \pi(k + 1/2) - i0} - \frac{1}{D(h_l) - \pi(k + 1/2) + i0}\right]\Big/2 = \right.$$

$$= \alpha_l^s + \pi i\beta_l^s\sum_{k=-\infty}^{\infty}\delta\left[D(h_l) - \pi\left(k + \frac{1}{2}\right)\right] +$$

$$\left. + Vp\left[\frac{1}{D(h_l) - \pi\left(k + \frac{1}{2}\right)} - \frac{1}{D(h_l) - \pi\left(k + \frac{1}{2}\right)}\right]\Big/2\right\} =$$

$$= \alpha_l^s + \pi i\beta_l^s\sum_{k=-\infty}^{\infty}\delta\left[D(h_l) - \pi\left(k + \frac{1}{2}\right)\right].$$

That is, imaginary medium pulse is originated. Imaginary velocity means flow rotation or oscillation; flow step is originated which will be destroyed in time $\Delta D(h_l) = \pi$ to originate repeatedly. Number of such steps is finite. But how to average this steps? You should pass to probabilistic interpretation of the description. That is, to average imaginary part over the period $D(h_l^k) - \pi\left(k + \frac{1}{2}\right)$. Then we have local complex average solution

$$\langle x_l \rangle - \alpha_l^s = \pi i \beta_l^s \left\langle \frac{\delta(h_l)}{\dot{D}_l(h_l)} \right\rangle \Big/ \pi = \frac{i\beta_l^s}{\dot{D}_l(h_l^k)};$$

$$D_l(h_l^k) = \pi\left(k + \frac{1}{2}\right).$$

Continuous part of the solution has positive and negative parts which are compensated when averaged. To obtain a global average value it is necessary to average with respect to value k, so we have

$$\langle x_l \rangle - \alpha_l^s = i\beta_l^s \sum_{k=-N}^{N} 1 \Big/ \dot{D}_l(h_l^k) \Big/ 2N.$$

We obtained complex velocity; imaginary part is defined up to multiplier. Real part of complex velocity corresponds to average value of velocity, and imaginary part is a mean square deviation. Simultaneously, there is a vortex motion consisting of positive and negative value of root from β_l^s.

Contribution of imaginary part to average value is equal to

$$\langle x_l \rangle = \alpha_l^s \pm i\sqrt{\beta_l^s}\gamma_l; \quad \gamma_l = \frac{1}{\dot{D}_l\left(h_l^k\right)}.$$

At that, module of average value, that defines real solution, is equal to

$$\left|\langle x_l \rangle\right| = \sqrt{(\alpha_l^s)^2 + \beta_l^s\gamma_l^2}; \quad \gamma_l = \frac{1}{\dot{D}_l\left(h_l^k\right)},$$

where balance position coordinates and time are non-dimensional, then, as we calculate square root of imaginary part, we define branch $\beta_l^s > 0$. Thus, average single-valued solution is found.

This multiplier γ_l depends on the surface roughness and it is found from numerical experiment. As numerical experiment has shown, for round smooth pipeline the multiplier is equal to $\gamma_l = 1$. At that, the smooth pipeline has a constant, minimum, average module of roughness inclination tangent equal to

$$\langle |\tan| \alpha \rangle = \frac{1}{R_{cr}} = \frac{1}{2300},$$

that is associated with molecular roughness, see article II section 1. For this, one term of series which determines flow velocity is used. We calculate this value for one term of the series for smooth surface. The solution is

$$x_l(t) = \alpha_l^s + \beta_l^s \tan(h_l),$$

where $D_l(h_l^k) = h_l^k = \pi(k + 1/2);$

$$\dot{D}_l(h_l^k) = 1;$$

$$\gamma_l = \left\langle \frac{1}{\dot{D}_l\left(h_l^k\right)} \right\rangle = 1.$$

This value exactly corresponds to experimental formula for round cross section pipeline if the solution roughness is taken into account. At that, to take roughness into account for internal problem, h_l^k is multiplied by $\left(\frac{1 + kR_{cr}/l}{2}\right)^{\sigma}$, here k/l is a constant average tangent of flowing surface inclination. From this we obtain $\gamma_l = \left(\frac{2}{1 + kR_{cr}/l}\right)^{\sigma}$, see article II section 1. At constant average roughness height, the coefficient γ_l is not a constant as k/l value is determined by other formula depending on dimensionless pressure, see article II section 1.