INSPECTION STOPPING RULES FOR CONVENTIONAL INSPECTION AND INSPECTION WITH MEMORY

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We have found the expected number of inspected items, before the inspection is stopped, for such inspection stopping rules as "out of the last r items, k items are defective" for both conventional inspection procedures and inspection with memory. Also, we have found the expected number of inspected items, before the inspection is stopped, for such an inspection stopping rule as "out of the last r_1 items, 2 items are defective, or out of the last r_2 items, k_2 items are defective" for conventional inspection.

Keywords: continuous inspection, inspection plan, probabilistic characteristics, recurrent events

The continuous inspection plan [1, 2, 3] is defined as a set of control rules and actions aimed at detecting the deterioration of product quality and at taking measures to normalize the production process. Those plans use such stopping rules as "out of the last r items, k items are defective" (r, k are integers, $r \geq 2$, $2 \leq k \leq r$). The application of such plans, when the inspection is stopped, implies that after the equipment is changed over or replaced, the inspection is recommenced without recording the results of the previous inspection.

This can be interpreted in the following way. Every produced item meets the standards with probability close to 1. Long-term production may lead to equipment failure, causing product quality deterioration. Once product quality deterioration is detected, inspection is stopped, the equipment is changed over or replaced to achieve product quality recovery and the inspection is recommenced without recording the results of the previous inspection.

Equipment changeover is impossible in human health risk management. In this case, when inspection is stopped, health risk reduction measures are taken. However, such measures cannot produce an immediate effect as they have a longer term impact. Taking this into account [4, 5] has proposed a new approach – continuous statistical inspection with memory. In contrast to the conventional inspection plan, continuous inspection with memory memorizes the last result after stopping of the inspection procedure and the next inspection does not start from "point zero" but recommences taking into account the previous inspection data.

Inspection stopping rules as recurrent events

It has been shown [4, 5] that an inspection stopping rule can be interpreted as an event E, which a finite set of conditions $A_1, A_2, \ldots, A_N$ correspond to, and the expected number of the inspected items until the occurrence of the event E is:

$$\mu(E) = \frac{\sum_{j=0}^{l-1} c_j}{P(E)}, \qquad (1)$$

where $A_1, A_2, \ldots, A_N$ are conditions corresponding to the event E; $l = \max_{1 \leq i \leq N} L(A_i)$ is the maximum length of a condition corresponding to the event E; c_h is the probability of transition from conditions $A_1, A_2, \ldots, A_N$, corresponding to the event E, into the same conditions by h steps.

Main results

Inspection stopping rules Π_1 are used for the plans of continuous inspection by attributes. The probability of each item to be non-defective and defective is p and $q = 1 - p$, respectively. A non-defective item will be hereinafter designated as "0", and a defective item as "1".

Let us consider a classical case of inspection, i.e. the case where the previous inspection data are not recorded. We list all the conditions, which cause the stopping of inspection, for the stopping rules "out of the last r items, k items are defective" with $k \geq 2$, $r \geq k$. These conditions are:

$$\left\langle \underbrace{1,1,\ldots,1}_{k} \right\rangle, \left\langle \underbrace{1,0,1,\ldots,1}_{k+1} \right\rangle, \left\langle \underbrace{1,1,0,1,\ldots,1}_{k+1} \right\rangle, \ldots;$$

$$\left\langle \underbrace{1,\ldots,1,0,1}_{k+1} \right\rangle, \left\langle \underbrace{1,0,0,1,\ldots,1}_{k+2} \right\rangle, \ldots, \left\langle \underbrace{1,\ldots,1,0,0,1}_{k+2} \right\rangle, \ldots; \left\langle \underbrace{1,\underbrace{0,\ldots,0}_{r-k},1,\ldots,1}_{r} \right\rangle, \ldots, \left\langle \underbrace{1,\ldots,1,\underbrace{0,\ldots,0}_{r-k},1}_{r} \right\rangle.$$

It is not difficult to see that the conditions, which have the length (the number of noughts and ones in a condition) of k are C_{k-2}^{k-2} conditions (i.e. only one condition), those, which have the length of $(k+1)$, are $C_{(k+1)-2}^{k-2}$ conditions. Using the same line of reasoning, we obtain that the conditions, which have the length of r are C_{r-2}^{k-2} conditions. Therefore, the probability of the event E_1, which denotes the stopping of inspection according to the rule "out of the last r items, k items are defective", is equal to

$$P(E_1) = q^k \sum_{i=k}^{r} p^{i-k} C_{i-2}^{k-2}$$

or

$$P(E_1) = q^k \sum_{j=0}^{r-k} p^j C_{j+k-2}^{j}.$$

Let l be the maximum length of the conditions corresponding to the event E_1, l equals r. Note that $c_0 = 1$ and let us find c_h, the probability of transition from the conditions, corresponding to the event E_1, into the same conditions (h varies from 1 to $l-1$).

We shall use the following line of reasoning. During h steps, there can be i defective items (i varies from 1 to $k-1$). The number of h steps should not be fewer than the number of defective items, but should not exceed the number, which equals $(l-k+i)$. Moreover, the number of variants of transition from the conditions, corresponding to the event E_1, into the same conditions is C_{h-1}^{i-1}. Therefore,

$$\sum_{j=1}^{l-1} c_j = \sum_{i=1}^{k-1} \sum_{h=i}^{l-(k-i)} C_{h-1}^{i-1} q^i p^{h-i}$$

or

$$\sum_{j=1}^{l-1} c_j = \sum_{i=1}^{k-1} \sum_{h=i}^{r-(k-i)} C_{h-1}^{i-1} q^i p^{h-i}.$$

Then

$$\sum_{j=0}^{l-1} c_j = 1 + \sum_{i=1}^{k-1} \sum_{h=i}^{r-(k-i)} C_{h-1}^{i-1} q^i p^{h-i}.$$

According to formula (1), we obtain the following equation for the expected number of the inspected items, before the inspection is stopped by the rule "out of the last r items, k items are defective" at $k \geq 2$, $r \geq k$, when each item has probability p of being non-defective, and probability $q = 1 - p$ of being defective:

$$\mu(E_1) = \frac{\sum_{j=0}^{l-1} c_j}{P(E_1)} = \frac{1 + \sum_{i=1}^{k-1} \sum_{j=i}^{r-k+i} q^i p^{j-i} C_{j-1}^{i-1}}{q^k \sum_{j=0}^{r-k} p^j C_{j+k-2}^{j}}. \quad (2)$$

In fact, we have proved the following theorem.

Theorem 1. The expected number of the inspected items, before the inspection is stopped in conventional inspection by the rule "out of the last r items, k items are defective", with q denoting the probability of defectiveness of each item and $p = 1 - q$ the probability of non-defectiveness of each item is equal to (2).

Let us consider inspection with memory, i.e. the case, where the data on the last inspected item is memorized when the inspection is stopped. Suppose that a defective item has been observed before the inspection is commenced. Let us list all the conditions (series), which lead to the stopping of the inspection, for such inspection stopping rules as "out of the last r items, k items are defective" at $k \geq 2$, $r \geq k$. These are the following conditions:

$$\langle \underbrace{1,1,\ldots,1}_{k-1} \rangle, \langle \underbrace{0,1,\ldots,1}_{k} \rangle, \langle \underbrace{1,0,1,\ldots,1}_{k} \rangle, \ldots;$$

$$\langle \underbrace{1,\ldots,1,0,1}_{k} \rangle, \langle \underbrace{1,0,0,1,\ldots,1}_{k+1} \rangle, \ldots, \langle \underbrace{1,\ldots,1,0,0,1}_{k+1} \rangle, \ldots;$$

$$\langle \underbrace{1,\underbrace{0,\ldots,0}_{r-k}, 1,\ldots,1}_{r-1} \rangle, \ldots, \langle \underbrace{1,\ldots,1,\underbrace{0,\ldots,0}_{r-k}, 1}_{r-1} \rangle;$$

$$\left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{1,\dots,1}_{k}\right\rangle,\ \left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{1,0,1,\dots,1}_{k+1}\right\rangle,\ \left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{1,0,1,\dots,1}_{k+1}\right\rangle,\dots,$$

$$\left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{1,\dots,1,0,1}_{k+1}\right\rangle,\ \left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{1,0,0,1,\dots,1}_{k+2}\right\rangle,\dots,$$

$$\left\langle \underbrace{0,\dots,0,}_{r-k+1}\underbrace{1,\dots,1,0,0,1}_{k+2}\right\rangle,\dots,\ \left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{\underbrace{0,\dots,0,1,\dots,1}_{r-k}}_{r}\right\rangle,\dots,\ \left\langle \underbrace{0,\dots,0,1}_{r-k+1},\underbrace{\dots,1,\underbrace{0,\dots,0,1}_{r-k}}_{r}\right\rangle.$$

It is not difficult to see that the conditions, which have the length of $k-1$ and the length of $(r-k+1)+(k)$ are $C^{k-2}_{(k)-2}$ conditions, those, which have the length of k and the length of $(r-k+1)+(k+1)$ are $C^{k-2}_{(k+1)-2}$ conditions. Using the same line of reasoning, we obtain that the conditions, which have the length of $r-1$ and the length of $(r-k+1)+(r)$, are $C^{k-2}_{(r)-2}$ conditions. Therefore, the probability of the event E_1^P, which denotes the stopping of inspection according to the rule "out of the last r items, k items are defective", in inspection with memory is equal to

$$P\left(E_1^P\right)=q^{k-1}\sum_{i=k}^{r}p^{i-k}C^{k-2}_{(i)-2}+q^{k}\sum_{i=k}^{r}p^{(r-k+1)+(i-k)}C^{k-2}_{(i)-2}$$

or

$$P\left(E_1^P\right)=q^{k-1}\sum_{j=0}^{r-k}p^{j}C^{j}_{(j)+k-2}+q^{k}\sum_{j=0}^{r-k}p^{(r-k+1)+j}C^{j}_{(j)+k-2}.$$

Next, let us assume that l^P is the maximum length of the conditions, corresponding to the event E_1^P, l^P equals $2r-k+1$. Note that $c_0=1$ and let us find c_h, the probability of transition from the conditions, corresponding to the event E_1^P, into the same conditions (h varies from 1 to l^P-1). We reason by analogy with the previous case. Let us schematically present, for the sake of illustration, the structure of steps for the number of inspection steps, which vary from 1 to $r-1$:

$$\left\langle \overbrace{\underbrace{X,X,X,X,X,X,}_{i\,\text{defectiveitems}}\ [1]}^{h\ \text{steps from 1 to } r-1}\right\rangle,$$

$$\left\langle \overbrace{\underbrace{0,0,\dots\dots,0,}_{r-k+1}\ \underbrace{X,X,X,X,X,[1]}_{k\ \text{defective items + nondefective items}}}^{h\ \text{steps from } r+1 \text{ to } 2r-k+1}\right\rangle,$$

where "1" is a defective item, "0" is a non-defective item and "X" is either a defective item or a non-defective one. Then

$$\sum_{j=1}^{l^P-1}c_j=\sum_{i=1}^{k-1}\sum_{h=i}^{r-(k-i)}C^{i-1}_{h-1}q^{i}p^{h-i}+\sum_{j=0}^{r-k}C^{j}_{j+k-2}q^{k}p^{(r-k+1)+j}.$$

Therefore,

$$\sum_{j=0}^{l^P-1}c_j=1+\sum_{i=1}^{k-1}\sum_{h=i}^{r-(k-i)}C^{i-1}_{h-1}q^{i}p^{h-i}+\sum_{j=0}^{r-k}C^{j}_{j+k-2}q^{k}p^{(r-k+1)+j}.$$

Plugging the obtained values of the probability of the event and the sum of the probabilities of transition of the conditions, corresponding to this event, into the same conditions, into formula (1), we obtain:

$$\mu\left(E_1^P\right)=\frac{1+\sum\limits_{i=1}^{k-1}\sum\limits_{j=i}^{r-(k-i)}C^{i-1}_{j-1}q^{i}p^{j-i}+\sum\limits_{j=0}^{r-k}C^{j}_{j+k-2}q^{k}p^{(r-k+1)+j}}{q^{k-1}\sum\limits_{j=0}^{r-k}p^{j}C^{j}_{j+k-2}+q^{k}\sum\limits_{j=0}^{r-k}p^{r-k+1+j}C^{j}_{j+k-2}}. \quad (3)$$

Let us develop a theorem based on the proposition, that was proven above.

Theorem 2. The expected number of the inspected items, before the inspection is stopped, in inspection with memory by the rule "out of the last r items, k items are defective", with q denoting the probability of defectiveness of each item and $p = 1 - q$ the probability of non-defectiveness of each item is equal to (3).

Evidently, for any fixed set of r, k and q, the following inequality is true:

$$\mu(E_1^P) < \mu(E_1). \tag{4}$$

An additional result

It is clear that we can similarly consider inspection stopping rules P_2, which are employed for the plans of conventional continuous inspection by attributes. The probability of each item to be non-defective and defective is supposed to be p and $q = 1 - p$, respectively. The inspection stopping rule P_2 is "out of the last r_1 items, k_1 items are defective, or out of the last r_2 items, k_2 items are defective", where $r_2 > r_1$, $k_2 > k_1$ and $k_1 > 1$.

Let us consider a conventional inspection case, i.e. the case where the inspection data is not recorded when the inspection is stopped. For inspection stopping rules P_2, assume that $k_2/(k_1 - 1)$ is an integer (note that at $k_1 = 2$ the number is always an integer) and let us describe all the conditions corresponding to the occurrence of the event E_{r_1,r_2,k_1,k_2}, i.e. the inspection stopping rule "out of the last r_1 items, k_1 items are defective, or out of the last r_2 items, k_2 items are defective".

Let us first list those conditions, which correspond to the occurrence of the event "out of the last r_1 items, k_1 items are defective". The conditions are:

$$\langle \underbrace{1,\dots,1,}_{k_1} \rangle \langle \underbrace{1,0,1,\dots,1}_{k_1+1} \rangle,\dots,\langle \underbrace{1,\dots,1,0,1}_{k_1+1} \rangle,\dots,\langle \underbrace{1,\underbrace{0,\dots,0,}_{r_1-k_1}1,\dots,1}_{r_1} \rangle,\dots,\langle \underbrace{1,\dots,1,\underbrace{0,\dots,0,}_{r_1-k_1}1}_{r_1} \rangle.$$

Then, we list the conditions, which correspond to the occurrence of the event "out of r_2 ($r_2 > r_1$) last items, k_2 ($k_2 > k_1$) items are defective". Note that until the occurrence of such an event, the condition, corresponding to this event, may have no more than $(k_1 - 1)$ successive defective items. Otherwise, the event "out of the last r_1 items, k_1 items are defective" would occur earlier. Moreover, between the neighboring groups consisting of i and j defective items, if $i + j \geq k_1$, there should be a group consisting of no fewer than $(k_1 - t)$ non-defective items, where $t = \max(i; j)$. Now, taking into account that $k_2/(k_1 - 1)$ is an integer, we assume that it equals k_0. Thus, the minimum length, which is equal to

$$n_0 = k_2 + (k_0 - 1)(r_1 - k_1 + 1) = k_2 + \left(\frac{k_2}{k_1 - 1} - 1\right)(r_1 - k_1 + 1),$$

belongs to the following condition

$$\langle \underbrace{1,\dots,1}_{k_1-1},\underbrace{0,\dots,0}_{r_1-k_1+1},\underbrace{1,\dots,1}_{k_1-1},\dots,\underbrace{1,\dots,1,}_{k_1-1} \rangle$$

where k_0 groups consisting of $(k_1 - 1)$ defective items and $(k_0 - 1)$ group consisting of $(r_1 - k_1 + 1)$ non-defective items. It is clear that the length, which is equal to

$$k_2 + \left(\frac{k_2}{k_1 - 1} - 1\right)(r_1 - k_1 + 1) + 1,$$

belongs to the following conditions:

$$\langle \underbrace{1,0,\dots,1}_{k_1},\underbrace{0,\dots,0}_{r_1-k_1+1},\underbrace{1,\dots,1}_{k_1-1},\dots,\underbrace{1,\dots,1,1}_{k_1-1} \rangle,\langle \underbrace{1,0,1,\dots,1,}_{k_1}\underbrace{0,\dots,0,}_{r_1-k_1+1}\underbrace{1,\dots1}_{k_1-1},\dots,\underbrace{1,\dots,1}_{k_1-1} \rangle,\dots,$$

$$\langle \underbrace{1,\dots,1,}_{k_1-1}\underbrace{0,\dots,0,}_{r_1-k_1+1}\underbrace{1,\dots,1}_{k_1-1},\dots\underbrace{1,\dots1,0,1}_{k_1} \rangle.$$

Using the same line of reasoning, we find that the maximum length, which is equal to r_2, belongs to such conditions as

$$\langle \underbrace{1,0,\dots,0,}_{r_2-n_0}\underbrace{1,\dots,1,}_{k_1-2}\underbrace{0,\dots,0,}_{r_1-k_1+1}\underbrace{1,\dots,1}_{k_1-1},\dots,\underbrace{1,\dots,1}_{k_1-1} \rangle,\dots,\langle \underbrace{1,\dots1,}_{k_1-1}\underbrace{0,\dots0,}_{r_1-k_1+1}\underbrace{1,\dots,1}_{k_1-1},\dots,\underbrace{1,\dots,1,}_{k_1-2}\underbrace{0,\dots0,1}_{r_2-n_0} \rangle.$$

Then, the probability of the occurrence of the event E_{r_1,r_2,k_1,k_2} equals the sum of probabilities of the conditions, which are described above:

$$P\left(E_{r_1,r_2,k_1,k_2}\right)=q^{k_1}\sum_{i=0}^{r_1-k_1} p^i C^i_{i+k_1-2}+q^{k_2}p^{(k_0-1)(r_1-k_1+1)}\sum_{i=0}^{r_2-(k_0-1)(r_1-k_1+1)-k_2} p^i C^i_{i+k_2-2}. \qquad (5)$$

Let us fix $k_1 = 2$ (k_0 is an integer) and find $P\left(E_{r_1,r_2,2,k_2}\right)$, i.e. the probability of the occurrence of the event $E_{r_1,r_2,2,k_2}$:

$$P\left(E_{r_1,r_2,2,k_2}\right)=q^2\sum_{i=0}^{r_1-2} p^i+q^{k_2}p^{(k_2-1)(r_1-1)}\sum_{i=0}^{r_2-r_1(k_2-1)-1} p^i C^i_{i+k_2-2}. \qquad (6)$$

Now we find $\sum_{j=0}^{l-1} c_j$. To do this, let us note that base "1" is included into any condition, corresponding to the event $E_{r_1,r_2,2,k_2}$and $c_0 = 1$. Then, it is easy to see that

$$\sum_{j=0}^{l-1} c_j=1+q\sum_{i=0}^{r_1-2} p^i+q^{k_2-1}p^{(k_2-1)(r_1-1)}\sum_{i=0}^{r_2-r_1(k_2-1)-1} p^i C^i_{i+k_2-2}+\sum_{j=1}^{k_2-2} q^j p^{j(r_1-1)}\sum_{i=0}^{r_2-r(k_{21}-1)-1} p^i C^i_{i+j-1}. \qquad (7)$$

Indeed, the first summand reflects the fact that $c_0 = 1$, the second and the third summand reflects the probability of transition into the conditions, corresponding to $E_{r_1,r_2,2,k_2}$ from base "1". The forth summand is the sum of the probabilities of transition into the conditions, corresponding to the event $E_{r_1,r_2,2,k_2}$ from bases, which are different from base "1".

Thus, we obtain the expected number of the inspected items, before the inspection is stopped, by such inspection stopping rule as "out of the last r_1 items, 2 items are defective, or out of the last r_2 items, k_2 items are defective":

$$\mu(E_{r_1,r_2,2,k_2})=\frac{1+q\sum_{i=0}^{r_1-2} p^i+q^{k_2-1}p^{(k_2-1)(r_1-1)}\sum_{i=0}^{r_2-r_1(k_2-1)-1} p^i C^i_{i+k_2-2}+\sum_{j=1}^{k_2-2} q^j p^{j(r_1-1)}\sum_{i=0}^{r_2-r_1(k_2-1)-1} p^i C^i_{i+j-1}}{q^2\sum_{i=0}^{r_1-2} p^i+q^{k_2}p^{(k_2-1)(r_1-1)}\sum_{i=0}^{r_2-r_1(k_2-1)-1} p^i C^i_{i+k_2-2}}. \qquad (8)$$

In fact, we have proved the following theorem.

Theorem 3. The expected number of the inspected items, before the inspection is stopped in conventional inspection by the rule "out of the last r_1 items, 2 items are defective, or out of the last r_2 items, k_2 items are defective", with q denoting the probability of defectiveness of each item and $p = 1 - q$ the probability of non-defectiveness of each item, is expressed by formula (8).

Conclusion

Inspection stopping rules play a significant role in a continuous inspection plan, in which they are included.

In practice, using any inspection stopping rules, the following measures are taken. If the really inspected number of items, before the inspection is stopped, is fewer than the expected number of the inspected items for fixed p (the probability of item's non-defectiveness, i.e. for the normal production of items), then measures are taken to normalize the production process. These may be the producing equipment changeover or replacement in flow-line production or preventive measures in human health management. If the really inspected number of items, before the inspection is stopped, is greater than or equal to the expected number of the inspected items for fixed p, then inspection is continued without taking any measures.

This can be interpreted in the following way: what happened, happened. In this case, in conventional inspection statistical data is not recorded while inspection with memory memorizes the last inspection step. It is the distinction between inspection with memory and conventional inspection.

In this work, we demonstrate that the number of inspected items, before the inspection is stopped, for any fixed set r, k, q and for a fixed inspection stopping rule in inspection with memory is fewer than that in conventional inspection, which follows from relation (4). In [5] a table for inspection stopping rule "out of the last r items, 2 items are defective" was presented as an example.

The main results obtained in this work for the inspection stopping rule "out of the last r items, k items are defective" at $k \geq 2$, $r \geq k$ for conventional inspection as well as for inspection with memory and for the rule "out of the last r_1 items, 2 items are defective, or out of the last r_2 items, k_2 items are defective" for conventional inspection, are formulated in theorems.

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