BILATERAL ESTIMATES FOR PROBLEM OF TWO-PHASE FILTRATION OF NONCOMPRESSIBLEFLUID

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Considering the using a method of fictitious area for the system of not evolutional type, which will be a model in filtering problem of two-phase incompressible fluid taking with capillary forces The received results allow simulating the processes of oil extraction with the use of production and forcing wells for water blockage of formation under test.

Keywords: two-phase filtration, processes of oil extraction

Let's consider the using a method of fictitious area for the system of not evolutional type, which will be a model for us in filtering problem of two-phase incompressible fluid taking with capillary forces.

Let D – a certain plain simply connected area with enough smooth interface. In the cylinder $D_T = \{D_x[0 < t \le T]\}$, with side surface $S = \{\gamma\times[0 < t \le T]\}$, there is being searched the solution of mixed Cauchy problem:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \dfrac{\partial u_2}{\partial t} = div\left(k\lambda_1 gradu_1\right) + f_1; \\ -\dfrac{\partial u_1}{\partial t} + \dfrac{\partial u_2}{\partial t} = div\left(k\lambda_2 gradu_2\right) + f_2; \\ u_1(x,0) - u_2(x,0) = \psi(x), \quad x \in D, \quad u_{i/s} = 0, \; i = 1, 2. \end{cases} \tag{1}$$

In (1) $k = k(x) > 0, f_i = f_i(x, t), \lambda_i = \text{const} > 0$, and necessary for justification of the method of fictitious areas additional requirements for input data of the problem (1) will be specified in the process of explanation. First of all we mention that if

$$2R = u_1 - u_2,$$
$$P = \lambda_1 u_1 - \lambda_2 u_2 \tag{2}$$

The initial problem (1) is decomposed into two independent problems:

$$\text{div}(k\text{grand } P) + f_1 + f_2 = 0, \quad P_{|s} = 0 \tag{3}$$

and

$$\frac{\partial R}{\partial t} = div\left(k\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} gradR\right) + \frac{\lambda_2 f_1 - \lambda_1 f_2}{2\left(\lambda_1+\lambda_2\right)},$$
$$2R(x, 0) = \psi(x), \quad R_{|s} = 0 \tag{4}$$

For all that the time t comes in the problem (3) as a parameter Therefore, justification of the method of fictitious areas at the differential level might be carried out for each of the problems (3), (4). Also, it is necessary to note that if in the initial problem (1) instead of uniform boundary condition of the first type there will be examined the uniform boundary condition of the second type, so instead of $P_{|s} = 0$ и $R_{|s} = 0$ in (3) and (4) accordingly we will have

$$\frac{\partial P}{\partial n_{|s}} = 0, \quad \frac{\partial R}{\partial n_{|s}} = 0. \tag{5}$$

For the problem (3), (4), if $k \in C\left(\bar{D}\right)$, $\psi \in \dot{W}_2^1\left(D\right)$, true estimates are:

$$\|P\|_{W_2^2(D_T)} \le C_1 \|f\|_{L_2(D_T)},$$
$$\|P\|_{W_2^{2,1}(D_T)} \le C_2\left(\|f\|_{L_2(D_T)} + \|\psi\|_{W_2^1(D)}\right). \tag{6}$$

In accordance with the method of fictitious areas, let's add the initial area D with a area D_1 up to the area $D_0 = D \cup D_1$, with boundary $\Gamma = \partial D_1 \cup \gamma$, $S^0 = \{\Gamma\times[0 < t \le T]\}$. In the compound area D_0 let's study additional problems:

$$div\left(k_\varepsilon gradP_\varepsilon\right) + f_1^\varepsilon + f_2^\varepsilon = 0, \; x \in D, \; \Delta P_\varepsilon = 0, \; x \in D_1;$$
$$P_\varepsilon = 0, \quad x \in S;$$
$$\frac{\partial R_\varepsilon}{\partial t} = div\left(k_\varepsilon \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} gradR_\varepsilon\right) + \frac{\lambda_2 f_1^\varepsilon - \lambda_1 f_2^\varepsilon}{2\left(\lambda_1+\lambda_2\right)}, \quad x \in D; \tag{7}$$
$$R_\varepsilon(x, 0) = 0{,}5\psi(x);$$

$$
\begin{aligned}
&P_\varepsilon = 0, \quad x \in S^0;\\
&\frac{\partial R_\varepsilon}{\partial t} - \Delta R_\varepsilon = 0;\\
&R_\varepsilon(x,0) = 0, \quad x \in D_1;\\
&R_\varepsilon(x,0) = 0, \quad (x,t) \in S^0.
\end{aligned} \tag{8}
$$

On the break curve S we lay down fitting conditions

$$
\begin{aligned}
&[P_\varepsilon]_{|s} = 0, \ [R_\varepsilon]_{|s} = 0,\\
&\frac{\partial R_\varepsilon}{\partial N}\Big|_{s^+} = \frac{Q}{\varepsilon}\frac{\partial R_\varepsilon}{\partial n}\Big|_{s^-}.
\end{aligned} \tag{9}
$$

$\varepsilon > 0$ – series expansion parameter, Q – parameter, possessing the value: $Q = 1$ or $Q = -1$,

$$
k, f_i^\varepsilon = \begin{cases} k(x),\ f_i(x,t), x \in D \\ Q\cdot\varepsilon^\alpha, 0,\ x \in D_1 \end{cases} \tag{10}
$$

For the problem (3), (4) $\alpha < 0$, and for the same problems, but with boundary conditions (5) $\alpha > 0$. Through $\frac{\partial}{\partial N}$ the normal derivative has been represented. Further $[g]_{|s} = [g]_{|s^+} - [g]_{|s^-}$, and the signs of minus and plus mean that counterpart is a limiting value by the tending of x to γ inside or outside of D. The auxiliary problems (6)-(8) have the transparent physical sense. The absolute permeability is small ($\alpha > 0$) or big ($\alpha < 0$) depending on the type of boundary condition of the initial problem in fictitious area. As regards the input data in the fictitious area D_1, R is an analog of capillary pressure and that's why equality R to zero means that in the fictitious area there is not only displacing phase. The fitting condition in (6), (7) means that for the transfer through γ (γ – line of factors' break) phase pressures and phase rates are continuous.

For the solution of the problem (8)-(10), (7), (9), (10) the true estimates are:

$$
\left\| R - \frac{1}{2}\left(R_\varepsilon^+ + R_\varepsilon^-\right) \right\|_{W_2^{2,1}(D_T)} \le C\varepsilon^2; \tag{11}
$$

$$
\left\| P - \frac{1}{2}\left(P_\varepsilon^+ + P_\varepsilon^-\right) \right\|_{W_2^{2,0}(D_T)} \le C\varepsilon^2. \tag{12}
$$

Where P_ε^+ и P_ε^-, R_ε^+ и R_ε^- correspond to the solution of the problems (8)-(10), (7), (9), (10) by $Q = 1$ and $Q = -1$. Further the problem (8)-(10) is called the problem I, problem (7), (9), (10) – II.

Now the solution of the problem I we will search in the form of power series on the parameter ε, $\alpha = -1$.

Let $B_1 = \sum_{m=0}^{\infty} \varepsilon^m V_m$ в D_T, $B_2 = \sum_{m=1}^{\infty} \varepsilon^m V_m$ в D_T^1.

Where we put $D_T^1 = \{D_1 \times [0 < t \le T]\}$ formally in the problem I, so then we will get the system relatively to V_m and W_m:

$$
\begin{cases}
\frac{\partial V_0}{\partial t} = div\left(k\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} gradV_k\right) + \frac{\lambda_2 f_1 - \lambda_1 f_2}{2(\lambda_1+\lambda_2)}, (x,t) \in D_T\\
V_0(x,0) = \psi(x), \ x \in D\\
V_0 = 0, \ (x,t) \in S
\end{cases} \tag{13}
$$

$$
\begin{cases}
\frac{\partial W_1}{\partial t} - \Delta W_1 = 0, \ (x,t) \in D_T^1\\
W_1(x,0) = 0, \ x \in D_1\\
\frac{\partial W_1}{\partial n} = q\frac{\partial V_0}{\partial n}, \ (x,t) \in S\\
W_1 = 0, \ (x,t) \in S^0
\end{cases}
$$

By $m \le 1$

$$
\begin{cases}
\frac{\partial V_m}{\partial t} = div\left(k\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} gradV_m\right) + \frac{\lambda_2 f_1 - \lambda_1 f_2}{2(\lambda_1+\lambda_2)}, (x,t) \in D_T\\
V_m(x,0) = \psi(x), \ x \in D\\
V_m = W_m, \ (x,t) \in S
\end{cases}
$$

$$\begin{cases} \dfrac{\partial W_{m+1}}{\partial t} - \Delta W_{m+1} = 0, \ (x,t) \in D_T^1 \\ W_{m+1}(x,0) = 0, \ x \in D_1 \\ \dfrac{\partial W_{m+1}}{\partial n} = q\dfrac{\partial V_m}{\partial n}, \ (x,t) \in S \\ W_{m+1} = 0, \ (x,t) \in S^0 \end{cases}$$

Let's suppose that functions in (13) are met the conditions of $V_m \in W_2^{2,1}(D_T)$, $k=0, 1, \ldots,$ $W_m \in W_2^{2,1}(D_T^1)$, $k = 1, 2, \ldots,$ so the following theorem is right.

Theorem 1. Let $f \in L_2(D_T)$, $\psi \in \dot{W}_2^1(D)$, so then ε_0 will be found this $0 < \varepsilon < \varepsilon_0$, that series B_1 and B_2 are absolutely converging in $W_2^1(D_T)$ and $W_2^1(D_T^1)$ and so correspondently the equalities are true:

$$\begin{aligned} R_\varepsilon &= B_1; \ (x,t) \in D_T; \\ R_\varepsilon &= B_2; \ (x,t) \in D_T^1. \end{aligned} \qquad (14)$$

Where R_ε – solution of the problem I.

Theorem proving. From the theory of uniform boundary problems and conditions of matching we have

$$\|W_m\|_{W_2^{2,1}(D_T^1)} \le C_1 \left\|\frac{\partial W_m}{\partial n}\right\|_{W_2^{\frac{1}{2},1}(s)} = C_1 \left\|\frac{\partial V_{m-1}}{\partial N}\right\|_{W_2^{\frac{1}{2},1}(s)} \le C_1 C_2 \|\partial V_{m-1}\|_{W_2^{2,1}(D_T)}, \qquad (15)$$

where constants C_1, C_2 depend on areas D, D_1 and factors of initial problem and don't depend on ε.

Applying the theory of trails in Sobolev spaces W_1^p

$$\|V_m\|_{W_2^{2,1}(D_T)} \le C_3 \|V_m\|_{W_2^{\frac{3}{2},1}(s)} = C_3 \|W_m\|_{W_2^{\frac{3}{2},1}(s)} \le C_3 C_4 \|W_{m-1}\|_{W_2^{2,1}(D_T^1)}.$$

Then from (6) and (15) it follows

$$\|V_m\|_{W_2^{2,1}(D_T)} \le C_5 \|V_{m-1}\|_{W_2^{2,1}(D_T)}, \ m \ge 1;$$

$$\|V_0\|_{W_2^{2,1}(D_T)} \le C\left(\|f\|_{L_2(D_T)} + \|\psi\|_{W_2^1(D)}\right), \qquad (16)$$

where $C_5 = C_1 \cdot C_2 \cdot C_3 \cdot C_4$.

Let $\varepsilon < \varepsilon_0 = C_5^{-1}$, thenseries$B_1$is absolutely converging in $W_2^{2,1}(D_T)$. For getting equalities (14) we multiply (13) by ε^m and sum on m, we have:

$$LB_1 = f, \ (x,t) \in D_T;$$

$$S_1(x,0) = \psi(x);$$

$$B_1 = B_2, \ (x,t) \in S;$$

$$\frac{\partial B_1}{\partial t} - \Delta B_2 = 0, \ (x,t) \in D_T^1;$$

$$\frac{\partial B_2}{\partial n} = Q\varepsilon \frac{\partial B_1}{\partial N}, \ (x,t) \in S;$$

$$B_2(x,0) = 0, \ x \in D_1;$$

$$B_2(x,t) = 0, \ (x,t) \in S_T^0. \qquad (17)$$

L – operator in the left part (13)

Si it follows that $R_\varepsilon = B_1$ in D_T, $R_\varepsilon = B_2$ in D_T^1 by $0 < \varepsilon < \varepsilon_0$.

From the theorem it follows unique existence of the solution of the additional problems (13), and the estimates are:

$$\|R - R_\varepsilon^+\|_{W_2^{2,1}(D_T)} \le C_6 \varepsilon \left(\|f\|_{L_2(D_T)} + \|\psi\|_{W_2^1(D)}\right);$$

$$\|R - R_\varepsilon^-\|_{W_2^{2,1}(D_T)} \le C_6' \varepsilon \left(\|f\|_{L_2(D_T)} + \|\psi\|_{W_2^1(D)}\right) \ldots (18)$$

Here it is $R_\varepsilon^+ = R_\varepsilon$ by $Q = 1$, $R_\varepsilon^- = R_\varepsilon$ by $Q = -1$, from absolute convergence of the series B_1 and B_2 it follows (11), let's bring it.

Theorem 2. If $0 < \varepsilon < \varepsilon_0$, so then the estimates are true

$$\left\|R - \frac{1}{2}\left(R_\varepsilon^+ + R_\varepsilon^-\right)\right\|_{W_2^{2,1}(D_T)} \le C_7 \varepsilon^2 \left(\|f\|_{L_2(D_T)} + \|\psi\|_{W_2^1(D)}\right). \qquad (19)$$

Where R – solution (4), R_ε^+, – solution (8) by $Q = 1$ and $Q = -1$, correspondently.

Theorem proving. From the theorem 1it follows the following expansion:

$$\begin{aligned} R_\varepsilon^+ &= \sum_{m=0}^{\infty} \varepsilon^m V_m^+, \ (x,t) \in D_T; \\ R_\varepsilon^+ &= \sum_{m=1}^{\infty} \varepsilon^m W_m^+, \ (x,t) \in D_T^1. \end{aligned} \qquad (20)$$

Here it is V_m^+, W_m^+ – solution (13) by $Q = 1$. Applying the theorem 1 for R_ε^- it is true:

$$\begin{aligned} R_\varepsilon^- &= \sum_{m=0}^{\infty} \varepsilon^m V_m^-, \ (x,t) \in D_T; \\ R_\varepsilon^- &= \sum_{m=1}^{\infty} \varepsilon^m W_m^-, \ (x,t) \in D_T^1. \end{aligned} \qquad (21)$$

Here it is V_m^-, W_m^- – solution (13) by $Q=-1$, it is easy to see that $V_0^+ \equiv V_0^- \equiv R$ – solutions (4).

Let $\overline{W}_1 = W_1^+ + W_1^-$, so the function $\overline{W}_1$ satisfies the following problem:

$$\frac{\partial \overline{W}_1}{\partial t} - \Delta \overline{W}_1 = 0,\ (x,t) \in D_T^1;$$

$$\frac{\partial \overline{W}_1}{\partial n_1} = 0,\ (x,t) \in S;$$

$$\overline{W}_1(x,0) = 0,\ (x,t) \in D_1;$$

$$\overline{W}_1 = 0,\ (x,t) \in S_T^0.$$

So from this we have

$$\overline{W}_1 = 0 \quad \text{or} \quad W_1^+ = -W_1^-.$$

Let's suppose that

$$\overline{V}_1 = V_1^+ + V_1^-,$$

so then the function $\overline{V}_1$ is the solution of the following problem:

$$L\overline{V}_1 = 0,\ (x,t) \in D_T;$$

$$\overline{V}_1(x,0) = 0,\ x \in D;$$

$$\overline{V}_1(x,t) = 0,\ (x,t) \in S.$$

From this we have

$$\overline{V}_1 = 0, \text{ then } V_1^+ = -V_1^-.$$

Even it supposes that

$$\overline{W}_2 = W_2^+ - W_2^-, \quad \overline{V}_2 = V_2^+ - V_2^-,$$

we will get

$$W_2^+ = W_2^-,\ V_2^+ = V_2^-.$$

Continuing this process by $m \geq 2$ we have:

$V_m^+ = V_m^-$, if m – even $V_m^+ = -V_m^-$, if m – uneven (22)

The from (21), using (19), (20), we will get in D_T

$$R_\varepsilon^+ = R + \varepsilon V_1^+ + \varepsilon^2 V_2^+ + \ldots;$$

$$R_\varepsilon^- = R - \varepsilon V_1^- + \varepsilon^2 V_2^- - \ldots \qquad (23)$$

Using the expansion (22) and estimates (17) we have come to (11):

$$\left\| R - \frac{1}{2}\left(R_\varepsilon^+ + R_\varepsilon^-\right) \right\| \leq \varepsilon^2 \left\| V_2^+ + \varepsilon^2 V_4^+ + \ldots \right\|_{W_2^{2,1}(D_T)} \leq C_8 \varepsilon^2 \left\| V_0^+ \right\|_{W_2^{2,1}(D_T)} \leq$$

$$\leq C_9 \varepsilon^2 \left(\|f\|_{L_2(D_T)} + \|\psi\|_{W_2^1(D_T)} \right).$$

Estimate (12) is got in much the same way.

Accuracy of received bilateral approximation in this case is limited by the estimate (11), (12). If only to get bilateral estimates R, P, with specified accuracy ε^P, we will use the idea of Richardson extrapolation.

Let's make extrapolated solutions $U_p^\pm$, being a linear combination $R_{\varepsilon_m}^\pm$, with some weight:

$$U_p^+ = \sum_{m=1}^{p} \beta_m R_{\varepsilon_m}^+;$$

$$U_p^- = \sum_{m=1}^{p} \beta_m R_{\varepsilon_m}^-,\ (x,t) \in D_T. \qquad (24)$$

Concrete view off coefficients β_m depends on choice of sequence $\varepsilon > \varepsilon_1 > \ldots > \varepsilon_p > 0$ and accuracy figure p. The more spread choice is:

$$\varepsilon_m = \frac{\varepsilon}{m},\quad m = 1, \ldots, p. \qquad (25)$$

By which coefficients β_m are in the explicit form

$$\beta_m = \frac{(-1)^{p-m} m^p}{m!(p-m)!},\quad m = 1, \ldots, p. \qquad (26)$$

And it satisfies the conditions

$$\sum_{m=1}^{p} \beta_m = 1;$$

$$\sum_{m=1}^{p} \frac{\beta_m}{m_j} = 0,\quad j = 1, \ldots, p-1.$$

By this way of the task ε_m, β_m we find that

$$U_p^+ = \sum_{m=1}^{p} \beta_m R_{\varepsilon_m}^+ = \sum_{m=1}^{p} \beta_m R + \sum_{m=1}^{p-1} \sum_{j=1}^{p} \beta_j \left(\frac{\varepsilon}{j}\right) V_m^+ + \sum_{m=1}^{p} \beta_m \left(\frac{\varepsilon}{m}\right)^p V_p^+ +$$

$$+O\left(\varepsilon^{p+1}\right) = R \sum_{m=1}^{p} \beta_m + \sum_{m=1}^{p-1} \varepsilon^m V_m^+ \sum_{j=1}^{p} \beta_j \left(\frac{1}{j}\right)^m + \varepsilon^p V_p^+ \cdot \sum_{j=1}^{p} \beta_j \left(\frac{1}{j}\right)^m V_p^+ +$$

$$+O\left(\varepsilon^{p+1}\right) = R + C_{10} \varepsilon^p V_p^+ + O\left(\varepsilon^{p+1}\right),$$

where $C_{10}=\sum_{j=1}^{p}\beta_j\left(\frac{1}{j}\right)^p$.

In much the same way it is

$$U_p^- = R - C_{10}\varepsilon^p V_p^+ + O\left(\varepsilon^{p+1}\right).$$

Let p – uneven, then $V_p^+ = -V_p^-$ and, it means,

$$U_p^+ = R + C_{10}\varepsilon^p V_p^+ + O\left(\varepsilon^{p+1}\right); \quad U_p^+ = R - C_{10}\varepsilon^p + O\left(\varepsilon^{p+1}\right). \tag{27}$$

With the help of this statement the theorem has been proved.

Theorem 3. Let $f \in L_2(D_T)$, R – solution (4), R_ε^+, R_ε^- – solution (8) corresponds the choice $Q = 1$ и $Q = -1$. So then for all $(x, t) \in D_T$ and $0 < \varepsilon < \varepsilon_0$ it has a place the asymptotic point wise bilateral inequality:

$$O\left(\varepsilon^{p+1}\right) + \min\left\{U_p^+, U_p\right\} \le R \le \max\left\{U_p^+, U_p^-\right\} + O\left(\varepsilon^{p+1}\right). \tag{28}$$

Where p – uneven, and U_p^+, U_p^- are defined on the formula (26).

The same estimates are received for the function P, and power (2) for the functions u_1, u_2.

The received results allow simulating the processes of oil extraction with the use of production and forcing wells for water blockage of formation under test.