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ABOUT EFFECT OF «SPLITTING» FOR THE DIFFERENTIAL OPERATOR OF THE FOURTH ORDER WITH THE SUMMABLE POTENTIAL

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In article the differential operator of the fourth order with summable potential is considered. It is received the asymptotics of the solutions of the given differential equation by introducing of the integral equation. For the equations of the second order the asymptotics of the solutions has been received by Vinokurov V.A. and Sadovnichii V.A. in 1998. Their technique on the equations of order above the second is not transferred.

Boundary conditions are picked up in such a manner that multiple «in the main» a series of eigenvalues «is split» on three unitary. Earlier (in 1997) this effect has been noticed by the author for the equation of the second order with discontinuous coefficients. It is received the asymptotics of eigenvalues of the considered differential operator. The deduced formulas it is enough for learning of the asymptotics of eigenfunctions and formulas of the first regularized trace of the studied operator.

All results of article are original and actual.

Key words: differential operators, asymptotics of eigenvalues, asymptotics of eigenfunctions, effect of «splitting», boundary conditions, summable potential

§1. Problem statement

Let's consider the differential operator of the fourth order with the summable potential, which is given by the differential equation

$$y^{(4)}(x)+q(x)y(x)=\lambda\cdot a^4\cdot y(x),\ \ 0\le x\le\pi,\ \ a>0, \tag{1}$$

and boundary conditions of a following kind:

$$y(0)=y''(0)=0;\quad y(\pi)=\alpha_0\cdot y\left(\frac{\pi}{2}\right);\quad y''(\pi)=\alpha_2\cdot y''\left(\frac{\pi}{2}\right), \tag{2}$$

$$\alpha_0+\alpha_2=4,\quad \alpha_0\in R,\quad \alpha_2\in R, \tag{3}$$

where λ is the spectral parameter, function $q(x)$ is called as potential.

We will assume that the potential is a summable function on a segment $[0;\pi]$:

$$q(x)\in L_1[0;\pi]\Leftrightarrow\left(\int_0^x q(t)dt\right)'_x=q(x)\ \text{ almost everywhere for }\ x\in[0;\pi] \tag{4}$$

Differential operators of a kind (1)–(2) with a condition (4) have started to be studied recently. Fundamental work [1] for the differential operator of the second order has appeared in the 2000 year. In it for the differential operator of the second order with summable potential it is received asymptotics of any order for solutions of the differential equation and asymptotics of eigenvalues of any order. In work as the [2] method which is distinct from a method of work [1], the differential operator of the fourth order in case of summable potential was studied. In the monograph [3] method of the work [2] is generalised on the differential operator of the 2n-th (n = 2, 3, 4, ...) order, on the functional-differential operators of the second and fourth order, and on the differential operators with discontinuous weight function.

The condition (3) provides so-called «effect of splitting» multiple in the main eigenvalues of the considered differential operator which in work [4] was considered by the author for the differential operator of the second order with discontinuous potential. As far as it is known to the author, for a case of differential operators of the fourth and higher order the given effect was not studied even for a case of the smooth potential.

§2. Asymptotics of the solutions of the differential equation (1) at $|s| \to +\infty$

For studying of asymptotics of eigenvalues of the boundary problems connected with the differential operator (1)–(3), it is necessary to know of the asymptotics of the solutions differential equation (1).

Let $\lambda = s^4$, $s = \sqrt[4]{\lambda}$ – some fixed branch of a root chosen by a condition $\sqrt[4]{1} = +1$. Let ω_k are the roots of the fourth degree from unit, that is

$$\omega_k^4 = 1, \quad \omega_k = \sqrt[4]{1} = e^{\frac{2\pi i(k-1)}{4}}; \quad k = 1,2,3,4, \quad \omega_1 = -\omega_3 = 1, \quad \omega_2 = -\omega_4 = i.$$

Numbers ω_k are on the unit circle and divide it into four equal parts.

In the work [2] we have proved the following theorem.

Theorem 1. The general solution of the differential equation (1) has the following expression:

$$y(x,s) = \sum_{k=1}^{4} C_k \cdot y_k(x,s), \qquad y^{(m)}(x,s) = \sum_{k=1}^{4} C_k \cdot y_k^{(m)}(x,s), \quad m = 0, 1, 2, 3, \tag{5}$$

where C_k (k =1,2,3,4) – arbitrary constants, $y_k(x, s)$ – linearly independent solutions of the differential equation (1), and for $|s| \to +\infty$ are valid following asymptotic decomposition:

$$y_k^{(m)}(x,s) = (as)^m \cdot \left\{ \omega_k^m \cdot e^{a\omega_k sx} - \frac{1}{4a^3 s^3} \cdot A_{3k}^m(x,s) + O\left(\frac{e^{\mathrm{Im}(|s|x)}}{s^6} \right) \right\}, \quad k = 1,2,3,4,$$

$$A_{3k}^m(x) = \sum_{n=1}^{4} \frac{\omega_n^m}{\omega_n^3} \cdot e^{a\omega_n sx} \cdot \int_0^x q(t) \cdot e^{a(\omega_k - \omega_n)st} \cdot dt_{qkn}, \quad m = 0,1,2,3. \tag{6}$$

Following formulas are thus true:

$$y_k(0;s)=1; \quad y_k^{(m)}(0;s)=(as)^m\cdot\omega_k^m; \quad k=1,2,3,4, \quad m=0,1,2,3. \tag{7}$$

An idea of decomposition of a kind (6)–(7) we have stated in the chapter 5 of the monograph [3].

The fundamental system of solutions $\{y_k(x,s)\}$, $k=1,2,3,4$, for the differential equation (1) is a generalization of the fundamental system of solutions for the differential operator of the second order, which is consisting from Josta´s functions.

§3. Studying of boundary conditions (2)

From boundary conditions (2) by means of formulas (5)–(7) it is received:

$$y(0)=0\Leftrightarrow\sum_{n=1}^{4}C_n\cdot y_n(0,s)=0\Leftrightarrow\sum_{k=1}^{4}C_n\cdot 1=0, \tag{8}$$

$$\frac{y''(0)}{(as)^2}=0\Leftrightarrow\sum_{n=1}^{4}C_n\cdot\frac{y_n''(0,s)}{(as)^2}=0\Leftrightarrow\sum_{n=1}^{4}C_n\cdot\omega_n^2=0. \tag{9}$$

The system of the equations (8)–(9) represents the system of two equations with four unknown variables which owing to conditions $\omega_1^2=\omega_3^2=1$, $\omega_2^2=\omega_4^2=-1$ has the unique solution

$$C_3=-C_1, \quad C_4=-C_2 \tag{10}$$

Substituting formulas (10) and (5)–(6) in the remained two boundary conditions from (2), we will receive system from two equations with two unknown variables which has nonzero solutions in that and only that case when the determinant of this system is equal to zero.

Therefore the following statement is true.

Theorem 2. The equation on the eigenvalues of differential operator (1)–(2)–(3) has the following expression:

$$f(s)=\begin{vmatrix} a_{11}-\dfrac{B_{311}(s)}{4a^3s^3}+O_1\left(\dfrac{1}{s^6}\right) & a_{12}-\dfrac{B_{312}(s)}{4a^3s^3}+O_2\left(\dfrac{1}{s^6}\right) \\ a_{21}-\dfrac{B_{321}(s)}{4a^3s^3}+O_3\left(\dfrac{1}{s^6}\right) & a_{22}-\dfrac{B_{322}(s)}{4a^3s^3}+O_4\left(\dfrac{1}{s^6}\right)\end{vmatrix}=0, \tag{11}$$

in which the following designations are entered:

$$a_{11}=z^{2\omega_1}-z^{-2\omega_1}-\alpha_0\cdot z^{\omega_1}+\alpha_0\cdot z^{-\omega_1}; \quad a_{12}=z^{2\omega_2}-z^{-2\omega_2}-\alpha_0\cdot z^{\omega_2}+\alpha_0\cdot z^{-\omega_2};$$

$$a_{21}=\omega_1^2\cdot[z^{2\omega_1}-z^{-2\omega_1}-\alpha_2\cdot z^{\omega_1}+\alpha_2\cdot z^{-\omega_1}]; \quad a_{22}=\omega_2^2\cdot[z^{2\omega_2}-z^{-2\omega_2}-\alpha_2\cdot z^{\omega_2}+\alpha_2\cdot z^{-\omega_2}];$$

$$B_{311}(s)=[A_{31}(\pi,s)-A_{33}(\pi,s)]-\alpha_0\cdot\left[A_{31}\left(\frac{\pi}{2},s\right)-A_{33}\left(\frac{\pi}{2},s\right)\right],$$

$$B_{312}(s)=[A_{32}(\pi,s)-A_{34}(\pi,s)]-\alpha_0\cdot\left[A_{32}\left(\frac{\pi}{2},s\right)-A_{34}\left(\frac{\pi}{2},s\right)\right],$$

$$B_{321}(s)=[A_{31}^2(\pi,s)-A_{33}^2(\pi,s)]-\alpha_2\cdot\left[A_{31}^2\left(\frac{\pi}{2},s\right)-A_{33}^2\left(\frac{\pi}{2},s\right)\right],$$

$$B_{322}(s)=[A_{32}^2(\pi,s)-A_{34}^2(\pi,s)]-\alpha_2\cdot\left[A_{32}^2\left(\frac{\pi}{2},s\right)-A_{34}^2\left(\frac{\pi}{2},s\right)\right]. \tag{12}$$

In formulas (11)–(12) we have entered a following designation:

$$z=e^{as\cdot\frac{\pi}{2}},\quad z^2=e^{as\pi}.$$

Expanding the determinant from (11) on columns for the sum of determinants, using properties of determinants, we receive:

$$f(s)=\{f_0(s)\}_{\text{осн}}-\frac{1}{4a^3s^3}\cdot B_3(s)+O\left(\frac{1}{s^6}\right)=0,$$

$$B_3(s)=\begin{vmatrix}B_{311} & a_{12}\\ B_{321} & a_{22}\end{vmatrix}+\begin{vmatrix}a_{11} & B_{312}\\ a_{21} & B_{322}\end{vmatrix}, \tag{13}$$

$$f_0(s)=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=\begin{vmatrix}z^{2\omega_1}-z^{-2\omega_1}-\alpha_0 z^{\omega_1}+\alpha_0 z^{-\omega_1} & z^{2\omega_2}-z^{-2\omega_2}-\alpha_0 z^{\omega_2}+\alpha_0 z^{-\omega_2}\\ z^{2\omega_1}-z^{-2\omega_1}-\alpha_2 z^{\omega_1}+\alpha_2 z^{-\omega_1} & -z^{2\omega_2}+z^{-2\omega_2}+\alpha_2 z^{\omega_2}-\alpha_2 z^{-\omega_2}\end{vmatrix}. \tag{14}$$

The equation «in the main approach» on the eigenvalues for the equation (13)–(14) is the equation

$$f_0(s)=0. \tag{15}$$

Expanding the determinant from (14)–(15) more detaily, it is possible to understand that the indicator diagram (that is the convex polygon constructed on powers of exponents, entering into the equation (15), see [3]), is a square, with tops $A(2;\ 2)$, $B(-2;\ 2)$, $C(-2;\ -2)$, $D(2;\ -2)$, and on each of the sides of this square there are two points. We will tell, on the side AB there are points $H(-1;\ 2)$ and $E(1;\ 2)$. Eigenvalues of the boundary problem (1)–(3) considered by us there are in four sectors of the infinitesimal angles which bisectors are middle perpendiculars to the sides of square $ABCD$.

Theorem 3. The equation on eigenvalues «in the main » of the boundary problem (1)–(2)–(3) for the side AB of a square $ABCD$ has the following appearance:

$$g_0(s) = z^{2\omega_1+2\omega_2} - \frac{\alpha_0+\alpha_2}{2}\cdot z^{\omega_1+2\omega_2} + \frac{\alpha_0+\alpha_2}{2}\cdot z^{-\omega_1+2\omega_2} - z^{-2\omega_1+2\omega_2} = 0. \qquad (16)$$

Substituting the condition (4) in the equation (16) we can divide on $z^{2\omega_2} \neq 0$ and by replacement $z^{\omega_1} = X$ reduce the equation (16) to the equation of the fourth degree $X^4 - 2X^3 + 2X - 1 = 0$, which has a simple root $X_4 = -1$ and a third-tuple root $X_1 = X_2 = X_3 = 1$.

Let's consider the sector connected with segment AB. We will leave in the equation (12) only the main terms on growth exponents.

Theorem 4. The equation on eigenvalues of the boundary problem (1)–(2)–(3) for the side BEHA of a square $ABCD$ has the following appearance:

$$g(s) = g_0(s) - \frac{1}{4a^3s^3}\cdot g_3(s) + O\left(\frac{1}{s^6}\right) = 0, \qquad (17)$$

and it is possible to write out $g_3(s)$ in an explicit form, considering formulas (11)–(13).

§4. Asymptotics of eigenvalues of the differential operator (1)–(3)

Let's consider at first a case simple «in the main» eigenvalue $X_4 = -1$. In this case we have:

$$X_4 = z^{\omega_1} = -1 \Leftrightarrow e^{a\omega_1 s\cdot\frac{\pi}{2}} = -1 \Leftrightarrow a\omega_1 s\cdot\frac{\pi}{2} = 2\pi ik + \pi i \Leftrightarrow s_{k,4} = \frac{(4k+2)\cdot i}{a}$$

– the basic approach for the side BEHA. Therefore the methods stated in the fifth chapter of the monograph [3], prove the following theorem.

Theorem 5. Asymptotics of eigenvalues of the boundary problem (1)–(2)–(3) in sector of segment BEHA of the indicator diagram has the following appearance:

$$s_{k,1} = \frac{iK_1}{a\omega_1} + \frac{id_{3k,4}}{a\omega_1\cdot K_1^3} + O\left(\frac{1}{K_1^6}\right), \quad K_1 = 4k+2, \quad \omega_1 = 1 \qquad (18)$$

and the coefficient $d_{3k,4}$ is equal

$$d_{3k,4} = \frac{1}{8\pi}\cdot\left[\int_0^{\pi} q(t)dt_{q11} - \int_0^{\pi} q(t)\cdot\cos\big((8k+4)t\big)dt_2 + 2\int_0^{\frac{\pi}{2}} q(t)dt_{q11} - 2\int_0^{\frac{\pi}{2}} q(t)\cdot\cos\big((8k+4)t\big)dt_2\right]. \qquad (19)$$

Asymptotics of the roots of the equation (17), considering formulas (16), (12)–(14), we will search by the method consecutive approaches of Horn. We will substitute the formula (18) in the equation (17) and we will equate coefficients by the same degrees K_1. We will come to a conclusion that the formula (19) is true.

Let's consider now a case of a third-tuple root $X_1 = X_2 = X_3 = 1$.

Theorem 6. Asymptotics of eigenvalues of the boundary problem (1)–(2)–(3) in sector of segment BEHA of the indicator diagram has the following kind:

$$s_{k,4,m} = \frac{4ki}{a\omega_1} + \frac{id_{1k,4,m}}{a\omega_1 \cdot k} + \frac{id_{2k,4,m}}{a\omega_1 \cdot k^2} + \frac{id_{3k,4,m}}{a\omega_1 \cdot k^3} + O\left(\frac{1}{k^4}\right), \tag{20}$$

and the factor $d_{1k,4,m}$ is equal

$$d_{1k,4,m} = -\frac{1}{4\pi} \cdot \sqrt[3]{2} \cdot e^{\frac{2\pi i}{3}\cdot(m-1)} \times$$
$$\times \sqrt[3]{\int_0^{\pi} q(t)dt - \int_0^{\pi} q(t)\cos(8kt)dt - 2\cdot\int_0^{\frac{\pi}{2}} q(t)dt + 2\cdot\int_0^{\frac{\pi}{2}} q(t)\cos(8kt)dt}, \tag{21}$$

where m = 1, 2, 3, that is we see that there was «splitting» multiple (third-tuple) «in the main» a root on three simple series of eigenvalues of the boundary problem (1)–(2)–(3),

$$d_{2k,4,m} = \frac{1}{96\pi^3}\cdot\frac{1}{d_{1k,4,m}}\cdot\left[\pi\cdot\int_0^{\pi} q(t)\sin(8kt)dt_4 - 2\cdot\int_0^{\pi} tq(t)\sin(8kt)dt_5 - \pi\cdot\left(\int_0^{\frac{\pi}{2}}\ldots\right)_4 + 4\cdot\left(\int_0^{\frac{\pi}{2}}\ldots\right)_5\right], \tag{22}$$

under the condition $d_{1k,4,m} \neq 0$, (and, if $d_{1k,4,m} = 0$, then $d_{1k,4,m} = d_{2k,4,m} = d_{3k,4,m} = \ldots = 0$, and all eigenvalues of the boundary problem (1)–(2)–(3) are found in an explicit form under the formula

$$s_{k,4,m} = \frac{4ki}{a\omega_1}), \tag{23}$$

$$d_{3k,4,m} = \frac{\pi^2}{48}\cdot d_{1k,4,m}^3 - \frac{d_{2k,4,m}^2}{d_{1k,4,m}} + \frac{1}{128\pi^3 d_{1k,4,m}} \times$$
$$\times\left[-\int_0^{\pi} q(t)dt_1 + \int_0^{\pi} q(t)\cos(8kt)dt_3 + 2\cdot\left(\int_0^{\frac{\pi}{2}}\ldots\right)_1 - 2\cdot\left(\int_0^{\frac{\pi}{2}}\ldots\right)_3\right] +$$
$$+\frac{1}{192\pi^3}\cdot\frac{1}{d_{1k,4,m}^2}\cdot\left[\pi^2\cdot d_{1k,4,m}^2\cdot\int_0^{\pi} q(t)dt_1 - 4\cdot d_{2k,4,m}\cdot\int_0^{\pi} tq(t)\sin(8kt)dt_5 + \ldots\right] \tag{24}$$

The remark. If instead of the differential equation (1) we will consider the equation

$$y^{(4)}(x)+p(x)y'(x)+q(x)y(x)=\lambda\cdot a^4\cdot y(x),\quad 0\le x\le\pi,\quad a>0,$$

with boundary conditions (2)–(3) (exactly such differential equation the author has considered in the work [2]), then the considered formulas (18) and (20) will be rewriten in a kind

$$s_{k,1}=\frac{iK_1}{a\omega_1}+\frac{id_{2k,4}}{a\omega_1\cdot K_1^2}+\frac{id_{3k,4}}{a\omega_1\cdot K_1^3}+O\left(\frac{1}{K_1^6}\right),\quad K_1=4k+2,\quad \omega_1=1,$$

$$s_{k,4,m}=\frac{4ki}{a\omega_1}+\frac{id_{1k,4,m}}{a\omega_1\cdot k^{\frac{2}{3}}}+\frac{id_{2k,4,m}}{a\omega_1\cdot k^{\frac{4}{3}}}+\frac{id_{3k,4,m}}{a\omega_1\cdot k^{\frac{6}{3}}}+O\left(\frac{1}{k^{\frac{8}{3}}}\right),\quad m=1,2,3$$

(that is there is «splitting» on fractional degrees).

If we will renumber the sectors connected with sides *AB*, *BC*, *CD*, *DA* of the square *ABCD* by numbers 1, 2, 3, 4 then the following theorem is fair. (Its proof is similar to the proof of theorems 5 and 6).

Theorem 7.

$$s_{k,n,m}=s_{k,1,m}\cdot z^{m-1}=s_{k,1,m}\cdot i^{m-1},\quad n=1,2,3,4. \tag{25}$$

The formulas similar to formulas (18)–(24), for boundary problems of type (1)–(2)–(3), can be received and for cases of differential operators of the sixth and eighth orders.

Formulas (18)–(24) it is enough for calculation of the first regularized trace of the differential operator (1)–(2)–(3).

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