SEMIFIELDS AND SHEAVES

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The basis of the theory of sheaf representations of semifields is stated in this work. The analogs of Pierce's and Lambek's sheaves from the theory of rings are defined for semirings. Functional characterizations of biregular semifields are obtained.

Introduction

The theory of semifields is a perspective area of a modern algebra, which we can examine either as a component of the theory of semirings or as groups with a complementary binary operation. Golan's monograph covers the theory of semirings [9]. Some questions of the theory of semifields were examined in articles [4], [5], [8].

The theory of semirings and the theory of semifields have been investigated by the participants of the scientific algebraic seminar in Vyatka State University of Humanities since 1994.

The basis of the theory of sheaf representations of semifields, which was initiated in [11]–[15], is stated briefly in this work. In this connection the class of biregular semifields is examined more completely.

Semifield is an algebraic structure, which is a multiplicative group and, at the same time, an additive commutative semigroup, and multiplication is distributive concerning addition from both sides. Semifields with added zero are the semirings with division, which are not rings. As rings and distributive lattices semifields with zero form an important class of semirings, which plays an essential role in a structural theory of semirings. *Idempotent* semifields (semifields with identity $u+u=u$) correspond lattice ordered groups. Semifields are related to rings, because every semifield has a ring of differences. Let's notice, that *cancellable* semifields (semifields with quasi-identity $u+w=v+w \Rightarrow u=v$) are embedded in their rings of differences.

When investigating semifields we can use functional method: studied semifield is realized as a semifield of sheaf's cross-sections of some semifields under the topological space. Many rings admit good functional (sheaf) representations [6], which in many things are transferred on semirings [7].

Let's introduce needed conceptions. The class of the unit of arbitrary congruence on semifield is called the *kernel* of semifield. The subset A of semifield U will be the kernel if and only if A is a normal subgroup of a multiplicative group U, satisfying condition: if $u, v \in U$, $u+v=1$, $a, b \in A$, then $ax+by \in A$. The lattice of all kernels (congruences) of semifield U is indicated Con$U$. The kernel of semifield U, generated by element u, is called *the main kernel* and indicated (u). The kernel $A \in \text{Con}U$ is called *complemented*, if there is the kernel $B \in \text{Con}U$, when $AB=U$ и $A \cap B=\{1\}$.

The kernel A of semifield U is called *finitely generated*, if $A=(u_1)\cdot\ldots\cdot(u_n)$ for the finite number of elements $u_1,\cdot\ldots,\cdot u_n \in U$. The semifield $U=(u)$ is called the *semifield with generator u*. The kernel (2) of semifield U, where $2=1+1$, is the smallest subsemifield in U, which is a kernel. If $U=(2)$, semifield U is called *bounded*. Semifield is called *reduced*, if the quasi-identity $u^2+v^2=uv+vu \Rightarrow u=v$ is executed in it. Semifield U is called *distributive* (*chain*, *simple*, *indecomposable*), if lattice Con$U$ is distributive (the lattice is a chain, two-element, has two complemented elements exactly).

The kernel P of semifield U is called *nonreducible*, if $A \cap B \subseteq P$ lead to $A \subseteq P$ or $B \subseteq P$ for every $A, B \in \text{Con}U$. Space Sp(U) of all nonreducible kernels of semifield , examined with stone topology, is called the *nonreducible spectrum* of semifield U. Its subspace

Max$U$, which consists of all maximal kernels, is called the *maximal spectrum* of semifield U. The *pseudocomplement* of kernel $A \in$ Con$U$ is the biggest kernel A^*, which gives {1} in meet with A. Let's assume $O_P=\{u \in U: \exists v \in U \backslash P\ (u) \cap (v)=\{1\}\}$ for arbitrary nonreducible kernel P of semifield U

About the properties of semifields

Let's formulate some general properties of semifields [12].

Property 1. *Every finitely generated semifield is semifield with generator.*

Property 2. *Every semifield is embedded in semifield with generator.*

Property 3. *Maximal kernels of any semifield are nonreducible.*

Property 4. *If U is semifield with generator, then spaces Sp(U) and MaxU are compact and every semifield U's own kernel is included in some its maximal kernel.*

Property 5. *The opposite thing is right for distributive semifields U: if space Sp(U) is compact, or space MaxU is compact and semifield U's own kernels are included in maximal kernels, then U is semifield with generator.*

Property 6. *The set of all complemented kernels of every semifield U generates Boolean sublattice of lattice ConU.*

Property 7. *If semifield U is either distributive or reduced and bounded, then ConU is lattice with pseudocomplements, and sets O_P, $P \in Sp(U)$, are kernels in U.*

The analog of Pierce's representation

Let's define the analog of Pierce's ring sheaf representation [1], [6, § 12] for any semifield U. Let's examine Boolean sublattice B(U) in Con$U$ of all semifield U's complemented kernels and space M(U) of all maximal ideals of Boolean lattice B(U) with stone topology. Disjunctive union Π of all factor semifields $U/\vee M$, where $M \in$ M(U), generates the structural sheaf of semifield U under zero-dimensional compact M(U). Let's indicate the semifield of all stalks of sheaf Π as Γ(M(U), Π).

Theorem 1. *Every semifield U is isomorphic to semifield Γ(M(U), Π) of sections of sheaf Π of factor semifields of semifield U under zero-dimensional compact M(U).*

Consequence 1. *Every semifield with final set of kernels is isomorphic to direct product of finite number of indecomposable semifields.*

Semifield U is called *strongly Gelfand*, if there is its complemented kernel A, that $A \subseteq M$ and $A \not\subset N$ for every two different maximal kernels M and N in U. The semifield, all kernels (main kernels) of which are complemented, is called *Boolean* (*biregular*). Biregular semifields are the analogs of biregular rings. It's clear, that biregular semifields are distributive.

If U is strongly Gelfand semifield, let's identify every maximal kernel $M \in$ Max$U$ with maximal ideal $\{A \in$ B(U): $A \subseteq M\}$ of Boolean lattice B(U). Then zero-dimensional compact M(U) will be the compactification of locally compact space Max$U$.

Proposition 1. *For strongly Gelfand semifield U space MaxU is embedded in M(U) homeomorphically.*

Theorem 2. *Any semifield U is biregular if and only if it is isomorphic to the semifield of all sections of some sheaf of simple and trivial semifields (U/$\vee$M) under zero-dimensional compact M(U).*

Consequence 2. *Every biregular semifield is factorized in direct product of biregular idempotent semifield and biregular bounded commutative semifield, which are specified explicitly accurate within isomorphism.*

Consequence 3. *In every biregular semifield every kernel is the meet of maximal kernels, and finitely generated kernels are main.*

In every biregular semifield U nonreducible kernels are maximal: Spec$U$=Max$U$. Semifields with generator correspond to rings with unity, their maximal spectrum is compact.

Theorem 3. *Following states are equivalent for every biregular semifield U:*

1) *MaxU is compact;*
2) *MaxU is homeomorphous to M(U);*
3) *U is the semifield with generator.*

Consequence 4. *Biregular semifield with generator is isomorphic to the semifield of all sections of Hausdorff sheaf of simple semifields under zero-dimensional compact, which is specified explicitly accurate within homeomorphism.*

Let's examine *pseudocomplement* $A^*=\{u\in U: (u)\cap A=\{1\}\}$ for kernel A of semifield U. In the case of distributive semifields U pseudocomplement A^* is the biggest kernel in U, the meet of which with this kernel is unit kernel $\{1\}$. Distributive semifield is called *Baer*, if the pseudocomplement of every its kernel is complemented.

Proposition 2. *Following states are right for kernel A of semifield Γ of all possible sections of the sheaf of simple semifields under zero-dimensional compact X, which has one generator:*

1) *A is complemented in Γ if and only if set* $\Delta A=\{x\in X: \forall s\in A\ s(x)=1\}$ *is closed-open in X;*

2) $A=B^*$ *for some kernel B in Γ if and only if ΔA is canonically closed in X, or it coincides with the closure of its interior.*

Topological space is called *extremally unconnected*, if the closure of any its open set is open again.

Theorem 4. *For biregular semifield with generator to be Baer, it's necessary and sufficient, that its maximal spectrum is extremally unconnected space.*

Theorem 5. *Semifield U is Boolean if and only if it is isomorphic to the semifield of all sections of the sheaf of simple and trivial semifields (U/∨M) under zero-dimensional compact M(U), the set of isolated points of which (MaxU) is dense everywhere.*

Consequence 5. *If semifield U is Boolean, space M(U) is the compactification of Stone-Cheh of a discrete space MaxU:* $M(U)\approx\beta MaxU$.

Consequence 6. *Boolean semifields with generator are direct products of the finite family of simple semifields accurate within isomorphism.*

Consequence 7. *Every Boolean semifield is isomorphic to direct product of Boolean idempotent semifield and finite direct product of reducible simple commutative semifield.*

The semifields of sections of compact sheaves

Let sheaf Π of semifields U_x under topological space X is given.

Let's assume for point $x\in X$:

$\Gamma^x=\{s\in\Gamma: s(x)=1=1_x\in U_x\}$ is the kernel of the semifield of sections $\Gamma=\Gamma(\Pi, X)$;

$\pi_x: \Gamma\to U_x$, $\pi_x(s)=s(x)$ для всех $s\in\Gamma$, is the homomorphism of semifields.

Sheaf Π is called a *compact sheaf*, if

1) X is compact;

2) Π is a *factor* sheaf, i.e. π_x is a surjective representation for every point $x\in X$;

3) $\Gamma^x\cdot\Gamma^y=\Gamma$ for every point $x\neq y$ from X.

Every compact sheaf Π has the following important **property:**

there is the cross-section $s\in\Gamma$ *with values 1 on Y и* $s(x)\notin P_x$ *for every closed set Y in X, point* $x\in X\setminus Y$ *and an nonreducible kernel* P_x of semifield U_x.

Proposition 3. *Every sheaf of semifields under zero-dimensional compact is compact.*

Proposition 4. *Kernels A and B of the semifield Γ(X, Π) of sections of any semifields' sheaf Π under zero-dimensional compact X are equal if and only if* $\pi_x(A)=\pi_x(B)$ *for all points* $x\in X$.

Consequence 8. *The lattice of kernels of the direct product of finite number of semifields is isomorphic to the direct product of lattices of actors' kernels.*

Theorem 6. *The maximal kernels of the semifield Γ(X, Π) of sections of semifields* U_x*'s compact sheaf are exactly kernels* $\pi_x^{-1}(K_x)$*, where* $x\in X$ *and* K_x *is a maximal kernel in* U_x*. If X is a zero-dimensional compact, it's also right for nonreducible kernels.*

Theorem 7. *The semifield Γ(X, Π) of sections of any sheaf Π of semifields* U_x *under zero-dimensional compact X is distributive (bounded) if and only if all its stalks* U_x *are distributive (bounded).*

The semifield is called *Gelfand*, if there are elements $a\in M\setminus N$ and $b\in N\setminus M$, where $(a)\cap(b)=\{1\}$, for every its unequal maximal kernels M and N. Maximal spectra of Gelfand semifields are Hausdorff, and every its nonreducible may be included into only one maximal kernel.

The semifield, which has the biggest own kernel, is called a *local* semifield. Biregular semifields and local semifields are strongly Gelfand, and strongly Gelfand semifields are Gelfand.

Theorem 8. *The following states are right for the semifield $\Gamma(X, \Pi)$ of cross-section of any compact sheaf of local semifields:*

1) *Γ is Gelfand;*

2) *if Γ is semifield with generator, being strongly Gelfand Γ is equivalent to the zero-dimensionality of compact X, and MaxΓ is homeomorphous to X.*

Theorem 6 and the state 2) of theorem 8 are sheaf variations of the classic theorem of Gelfand-Kolmgoroff about the rings of continuous functions.

The analog of Lambek's representation

The representation of Lambek [2], [5, § 11] is used in the theory of rings too. Let's spread this construction on semifields.

Let's assume that semifield U is distributive or reduced bounded semifield. Then sets O_P, $P\in$Spec$U$, will be kernels. Let's examine the family (O_P) of kernels of semifield U, which is indexed by points P of topological space Sp(U). It is an *open family of kernels*, i.e. the set $\{P\in \mathrm{Sp}(U)\colon u\in O_P\}$ is open in an nonreducible spectrum Sp(U), when every $u\in U$.

In reality,

$$\{P\in \mathrm{Sp}(U)\colon u\in O_P\}=\{P\in \mathrm{Sp}(U)\colon (u)^*\not\subseteq P\}=\mathrm{D}((u)).$$

That's why [3] there is the sheaf $\Pi=\Pi(U)$ of factor semifields U/O_P of semifield U under compact T_0-space Sp(U). It's *the structural sheaf* of semifield U, which is analogous to the sheaf of Lambek for rings [5].

Theorem 9. *Distributive semifield U with generator is strongly Gelfand if and only if it's isomorphic to the semifield $\Gamma(X, \Pi)$ of sections of sheaf Π of local semifields U_x under zero-dimensional compact X.*

There we can take a maximal spectrum Max$U$ as X, and factor semifields U/O_M as stalks U_M.

Theorem 10. *The semifield is biregular if and only if it's isomorphic to the semifield of all sections with the compact carriers of Hausdorff sheaf of simple semifields under a zero-dimensional locally compact space.*

Consequence 9. *For semifield to be biregular semifield with generator, it's necessary and sufficient, that it is isomorphic to the semifield of all sections of Hausdorff sheaf of simple semifields under zero-dimensional compact.*

The representation of reduced bounded semifields

Every reducible semifield U embeds in its ring of differences R=R(U). Every bounded semifield U is reducible and the lattice Con$U$ of its kernels is canonically isomorphic to the lattice IdR of all ideals of the ring of differences R [10]. Following representations $\gamma\colon \mathrm{Id}R\to \mathrm{Con}U$ and $\delta\colon \mathrm{Con}U\to \mathrm{Id}R$ determine the isomorphism of lattices IdR and Con$U$:

$$\gamma(I)=(I+1)\cap U \text{ for all } I\in \mathrm{Id}R,$$
$$\delta(A)=(A-1)U \text{ for all } A\in \mathrm{Con}U.$$

Meanwhile $\gamma(I\cap J)=\gamma(I)\cap\gamma(J)$ и $\gamma(I+J)=\gamma(I)\cdot\gamma(J)$ for every I, $J\in$IdR.

To be reduced for reducible semifield U is equivalent to be reduced for its ring of differences R, i.e. to be absent in R nonzero nilpotent elements. Reduced rings have the feature of symmetric property: if the multiplication of several elements of ring, taken is some order, is 0, the multiplication of these elements, taken in any other order, is 0 too.

Proposition 5. *The following states are equivalent for every nonreducible ideal Q of reduced ring T with unit:*

1) Q is a minimal nonreducible ideal;
2) Q is a minimal prime ideal,
3) $Q=O_Q$.

Then U is an arbitrary ***reduced bounded semifield*** and R is its ring of differences. Isomorphisms δ and γ retain irreducibility and keep kernels and ideals – the elements of lattices Con$U$ и IdR, respectively – finitely generated. Let's notice, that finitely generated kernels (ideals) of semifield U (ring R) are exactly compact elements of algebraic lattice Con$U$ (IdR) []. The following states are right for every A, $B\in$Con$U$, I, $J\in$IdR, u, $v\in U$:

$$A\cap B=\{1\} \Leftrightarrow \delta(A)\cap\delta(B)=\{0\}=0 \Leftrightarrow \delta(A)\cdot\delta(B)=0; \quad (1)$$
$$I^*=\mathrm{Ann}I=\{r\in R\colon rI=\{0\}\} \text{ and } \delta(A^*)=\mathrm{Ann}\delta(A); \quad (2)$$
$$\mathrm{Ann}(I+J)=\mathrm{Ann}I\cap\mathrm{Ann}J \text{ and } (AB)^*=A^*\cap B^*; \quad (3)$$
$$\delta((u))=R(u-1)R \textit{ is the main ideal of ring } R; \quad (4)$$
$$(u)\cap(v)=\{1\} \Leftrightarrow (u-1)(v-1)=0 \Leftrightarrow uv+1=u+v; \quad (5)$$
$$\delta(O_P)=O_{\delta(P)}. \quad (6)$$

Proposition 6. *If every finitely generated kernel of semifield U is the main kernel, all finitely generated ideals of its ring of differences are main.*

Proposition 7. *ConU is lattice with pseudocomplements, for every nonreducible kernel P of semifield U set O_P is kernel in U. Furthermore,*

$$\cap\{O_P\colon P\in\mathrm{Sp}(U)\}=\{1\}. \quad (7)$$

Proposition 8. *The minimum of nonreducible kernel P of semifield U is equivalent to equality $P=O_P$.*

Let's assume, that $\Gamma=\Gamma(\mathrm{Sp}(U), \Pi)$ is the semifield of all sections of sheaf Π with point-wise determined operations of addition and multiplication. We would remind you that the section of sheaf Π is every continuous representation s: $\mathrm{Sp}(U)\to\Pi$, where $s(P)\in U/O_P$ *for every* $P\in\mathrm{Sp}(U)$. Let's set the representation α: $U\to\Gamma$ by the formula

$$\alpha(u)(P)=uO_P\in U/O_P \text{ for all } P\in\mathrm{Sp}(U).$$

It's clear, that α will be the homomorphism of semifields, and because of (7) – the homomorphic embedding of semifield U in the semifield of sections Γ. The homomorphism α is required functional representation of this semifield U by the sections of sheaf Π.

There is isomorphic Lambek's sheaf representation ^ of reduced ring of differences R under *prime spectrum* Spec$R$. The functional representation α': $R\to\Gamma(\mathrm{Sp}(R), \Pi')$ is built as for semifield U. Factor rings R/O_Q of ring R for arbitrary nonreducible ideal Q in R are the layers of sheaf Π'. There is $\alpha'(r)=r^\wedge$ on $\mathrm{Spec}R\subseteq\mathrm{Sp}(R)$ for $r\in R$.

Proposition 9. *Representation α′ is isomorphism.*

Proposition 10. *Factor ring $R/O_{\delta(P)}$ is the ring of differences of factor semifield U/O_P for every nonreducible kernel P of semifield U, semifield is reduced, bounded and $O_{P/OP}=\{1\}$.*

Proposition 11. *Every reduced bounded semifield U is isomorphic to "dense" sub-semifield of semifield Γ(SpecU, Π) of sections of the sheaf Π of fac-*

tor semifields U/O_P under nonreducible spectrum SpecU.

Theorem 11. *Commutative semifield is isomorphic to the semifield of all sections of the sheaf of chain bounded commutative semifields under zero-dimensional compact if and only if it is strongly Gelfand distributive, reduced and bounded.*

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