a union of three half-lines, all of them centering in 0 point and making angles of 120^0.

3. As $C\left((\bar{z}+z+i)^2+1,\infty\right)=\left\{t^2+2ti \mid t\in \bar{R}\right\}$ is a parabola of the second order, then every of the sets, $C\left(\frac{\bar{z}}{z}\left((\bar{z}^2+z^2+i)^2+1\right),\infty\right)$ and $C\left((\bar{z}^2+z^2+i)^2+1+\frac{\bar{z}}{z},\infty\right)$, is a union of two parabolas.

In the conclusion of the article let us show some simple upper estimate of the number of polynomial lines, making up $C(F(z),a)$, where $a\in \overline{\mathbf{C}}$ is isolated singular l-point of the p.a. function $F(z)$.

Theorem 2

For any p.a. function $F(z)$ of the proximate poly-analyticity order $n\in N$, $n\geq 2$, and for its every isolated singular l-point $a\in \overline{\mathbf{C}}$ the set of all the elements from $C(F(z),a)\ \mathbf{I}\ \mathbf{C}$ can be represented in the form of a union of finite number of nontrivial polynomial lines, the quantity of which $l\in N$ satisfies the following conditions:

a). $l\leq 4(n-1)$;

б). $l\leq (n-1)!$ (with $n=2$ and with $n=3$ this estimate is exact).

The deduction of the theorem 2 is in [6].

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CHRACTERISTIC PROPERTIES OF SEQUENCES GENERATING FINITE ELEMENTS OF POLYANALYTIC FUNCTIONS' CLUSTER SETS IN THEIR ISOLATED SINGULAR L-POINTS

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1. For every poly-analytic function (p.a. function) [1-4]

$$F(z)=\sum_{k=0}^{n-1} f_k(z)\bar{z}^k,\ n\in N, \qquad (1)$$

set in a deleted neighborhood $\overset{0}{O}(a)$ of its isolated nonessential singularity $a\in\overline{\mathbf{C}}$, the representation $F(z)$ in $\overset{0}{O}(a)$ is possible in the following form [4, 6]:

$$F(z)=F_\infty(z)+F_w(z)+F_0(z), \qquad (2)$$

where p.a. functions $F_\infty(z)$, $F_w(z)$ and $F_0(z)$ are identically determined [4, 6] by expansion coefficient of analytic in $\overset{0}{O}(a)$ functions $f_k(z)$ $(k=0,\ldots,n-1)$ in a Laurent series. Functions $F_\infty(z)$, $F_w(z)$ and $F_0(z)$ allow to describe identically a cluster set [1, 2, 4, 6] $C(F(z),a)$ of a poly-analytic function $F(z)$ in a point $a\in\overline{\mathbf{C}}$, in particular, to establish if the a point is an isolated l-point for $F(z)$; [4, 6], without loss of reasoning generality, being always possible to pass on from studying the behaviour of an arbitrary p.a. function $F(z)$ in a deleted neighborhood $\overset{0}{O}(a)$ of its isolated singular l-point $a\in\overline{\mathbf{C}}$ to studying of the behaviour of a poly-analytic polynomial $P(z,\bar{z})\in\mathbf{C}[z,\bar{z}]\setminus\mathbf{C}$, i.e.

$$P(z,\bar{z})=p_0(z)+p_1(z)\bar{z}+\ldots+p_{m-1}(z)\bar{z}^{m-1}, \qquad (2)$$

$(p_j(z)\in\mathbf{C}[z];\ j=0,\ldots,m-1)$, in the neighbourhood of ∞ point; besides

$$C(F(z),a)=\bigcup_{j=1}^{m-1}(C(P(z,\bar{z}),\infty)+c_j),$$

where $c_j\in\mathbf{C}$ $(j=1,\ldots,m-1)$, $0<m<n$, $m\in N$.

In the clauses [3, 4] it was got the criterion of availability of the not identical to the constant p.a. polynomial (2) of finite limited values in ∞ point; the speciality of getting this criterion being that, that allows describing all the sequences generating all finite limited values in ∞ point of the p.a. polynomial; and it means it allows describing all the sequences generating all the finite limited values of an arbitrary p.a. function in its any isolated singular l-point.

2. To formulate the corresponding results, let us enter some auxiliary notions; besides let us suppose further that for any element $l\in\overline{\mathbf{C}}$ any sequence of complex numbers convergent to l is considered to be known.

Definition 1

About the sequence of complex numbers (z_n) we shall speak that it is made up of the sequences

$$(z_n^{(1)}),(z_n^{(2)}),\ldots,(z_n^{(s)}), \qquad (3)$$

if every component of any of the sequences of the set (3) is the component (and the only one component) of the sequence (z_n), and there are no other components in this sequence.

Thus, either any two components of different sequences from (3) or two any components of different numbers of the same sequence from (3)will turn out to be different components of the sequence (z_n); the values of the components of these sequences being not considered.

Now, taking into account that any sequence of complex numbers has a convergent subsequence (to the finite element from $\overline{\mathbf{C}}$ or to ∞) and that any sequence (z_n) possessing the property $p(z_n)\to 0$ where $p(z)$ is some not identical to the constant analytic polynomial from z, is limited, and also that this polynomial $p(z)$ has a nonempty finite collection of roots in $\overline{\mathbf{C}}$, it is possible to formulate the following lemma.

Lemma 1

For any not identical to the constant analytical polynomial $p(z)$ the sequence of complex numbers (z_n) possesses the property $p(z_n)\to 0$ when and only when it is made up of an arbitrary part of the finite set of any sequences, every one of which converges to one of the roots of the polynomial $p(z)$.

It is relevant, in addition to the notion of the composed sequence (or even just as another way to describe resembling situations), to formulate the notion of the sequence convergent to a finite set and the corresponding property (lemma 2).

Definition 2

About the sequence of complex numbers (z_n) we shall speak that it converges to a nonempty infinite set $M=\{m_1,m_2,...,m_k\}\subset\mathbf{C}$, if for any set $O_e(M)=\bigcup_{s=1}^{k}O_e(m_s)$ where $O_e(m_s)$ is any e-neighbourhood of the point m_s $(s=1,...,k)$, there is such a number $n_0(e)$ that for any natural n, if $n>n_0(e)$, then $z_n\in O_e(M)$; if the sequence (z_n) converges to M, but doesn't converge to any of its eigenparts, then we shall speak about the sequence (z_n), that it converges to all the set M.

The following lemma 2 is evident.

Lemma 2

Any sequence of complex numbers (z_n) possesses the following property $p(z_n)\to 0$, where $p(z)$ is some not identical to the constant analytical polynomial, when and only when it converges to the set (not obligatory to the whole one) of all complex roots of the polynomial $p(z)$.

The fact, that the notions of the composed sequence and the sequence convergent to a set, is directly relevant to establishing the characteristic property of sequences generating finite limited values of p.a. polynomials in ∞ point, will be evident after establishing the following fact (theorem 1), that, however, is necessary to precede with some agreements about the nomenclature.

3. Let $P(z,\bar z)$ be an arbitrary p.a. polynomial of the form (2) and not identical to the constant, and let

$$p_j(z)=c_0^{(j)}+c_1^{(j)}z+...+c_{k_j}^{(j)}z^{k_j},\ (j=0,...,m-1)$$

In addition to it let us consider that $P(z,\bar z)$ informally depends both on z and on $\bar z$, i.e. $P(z,\bar z)\in\mathbf{C}[z,\bar z]$, but $P(z,\bar z)\notin\mathbf{C}[z]$ and $P(z,\bar z)\notin\mathbf{C}[\bar z]$.

Let us mention that it won't violate the reasoning generality, as for any not identical to the constant polynomial from the ring $\mathbf{C}[z]$ or from the ring $\mathbf{C}[\bar z]$, its cluster set in ∞ point is settled with one ∞ element.

Let it further be that $s=\deg_{z,\bar z}P(z,\bar z)\in N$.

Besides (for the nomenclature brevity sake) let us agree to add formal summands with trivial coefficients to any from the polynomials $p_j(z)$ as far as possible, and let us consider from now that $k_j+j=s\ (j=0,...,m-1)$, but at least one of the coefficients $c_{k_j}^{(j)}$ is nonzero.

Let us denote now a p.a. polynomial composed of all monomials of the p.a. polynomial (2) $P(z,\bar{z})$ of the form $c_{k_j}^{(j)} z^{k_j} \bar{z}^{s-k_j}$, $(j=0,\dots,m-1)$ by the symbol $\tilde{P}(z,\bar{z})$, i.e.

$$\tilde{P}(z,\bar{z}) = \sum_{j=0}^{m-1} c_{k_j}^{(j)} z^{k_j} z^{s-k_j} .$$

It is evident that

$$\tilde{P}(z,\bar{z}) = z^s \cdot \sum_{j=0}^{m-1} c_{k_j}^{(j)} \left(\frac{\bar{z}}{z}\right)^{s-k_j} = z^s \cdot P\left(\frac{\bar{z}}{z}\right),$$

where $P(w)$ is a not identical to the constant analytical polynomial from w, then the following statement makes sense.

Theorem 1

Every sequence of complex numbers (z_n) generating a finite point of a cluster set in ∞ point of the not identical to the constant p.a. polynomial $P(z,\bar{z})$ possesses the following property: the sequence $\left(\frac{\bar{z}_n}{z_n}\right)$ converges to the set of all roots of a unit module of the analytic polynomial $P(w) \in \mathbf{C}[w] \setminus \mathbf{C}$.

Deduction

Let $P(z,\bar{z})$ be a p.a. polynomial differing the not identical to the constant one, and let a sequence of complex numbers (z_n) possesses properties: $z_n \to \infty$, and $(P(z_n,\bar{z}_n))$ converges to the finite limit, but then $\frac{P(z_n,\bar{z}_n)}{z_n^s} \to 0$, but $\left|\frac{\bar{z}_n}{z_n}\right| \equiv 1$, that proves the theorem.

Complementation

If a polynomial $P(w)$ constructed not for the identical to the constant p.a. polynomial $P(z,\bar{z})$ has no roots of a unit module, then $C(P(z,\bar{z}),\infty) = \{\infty\}$.

Theorem 1 allows concluding that every sequence of complex numbers (z_n) generating a finite limited value of the not identical to the constant p.a. polynomial in ∞ point is made up of a part of a finite set of sequences (their own for every (z_n)) $\left(z_n^{(j)}\right)$ of which every one converges to ∞, simultaneously "ranging" in the direction set by a straight line from the finite collection (for every $\left(z_n^{(j)}\right)$ a straight line from this collection is its own and it is possible to consider that it passes through the point $z=0$). The "structure" of every of the sequences $\left(z_n^{(j)}\right)$ was studied in [3] and [4], that allows formulating the following theorem; the situation for b.a. functions is fully considered in [3, 7].

Theorem 2

For any p.a. function $F(z)$ of poly-analyticity order $n \geq 2$ and its any isolated singular l-point $a \in \overline{\mathbf{C}}$ there is a finite collection of expressions of the following form:

$$z^{(j)}(x,t) = d_j + c_j\left(b_j + a_j\left(x + i \cdot \frac{t - P_j(x)}{x^{d_j}}\right)^{l_j}\right) \quad (j=1,\dots,s;\ s \leq n-1), \qquad (4)$$

where complex numbers $d_j, c_j, b_j, a_j\ \ (c_j, a_j \neq 0)$ as well as the coefficients of the polynomials $P_j(x)$ from the ring $R[x]$, are fully determined by the coefficients of analytic components of the p.a. function $F_\infty(z)$ corresponding to the p.a. function $F(z)$ and the point $a \in \overline{\mathbf{C}}$, besides $l_j, d_j \in N$, $\deg P_j(x) < d_j$ $(j=1,\dots,s;\ s \leq n-1)$; this collection of expressions (4) possessing the following properties:

1) For any finite limited value l in the point a of this p.a. function $F(z)$ any generating this point sequence (z_n) is made up of a part of the collection of sequences $\left(z^{(j)}\left(x_n^{(j)},t_n^{(j)}\right)\right)$ with $a=\infty$, or

$$\left(\frac{1}{z^{(j)}\left(x_n^{(j)},t_n^{(j)}\right)}+a\right) \text{ with } a\in \mathbf{C}, \qquad (5)$$

being obtained by substitution into the expressions (4) of some convergent real sequences $\left(x_n^{(j)}\right)$ and $\left(t_n^{(j)}\right)$; $\left(t_n^{(j)}\right)$ converging to some finite , and $\left(x_n^{(j)}\right)$ - to infinite limits;

2) Vice versa: for any convergent real-valued sequences (x_n) and (t_n), where (t_n) converges to an arbitrary finite, and (x_n) - to infinite limits, every of the sequences (5) generates a finite limited value of the p.a. function $F(z)$ in its isolated singular l-point $a\in \overline{\mathbf{C}}$ (with $a=\infty$ or $a\in \mathbf{C}$ accordingly).

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CLUSTER SETS OF BIANALYTIC FUNCTIONS IN THEIR ISOLATED SINGULARITIES

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Brief
General methods of finding an explicit cluster set of any bi-analytic function in its arbitrary isolated singularity.

1. Let it be required to find a cluster set $C(F(z),a)$ [1-6] of an arbitrary bi-analytic function (b.a. function) [2-6]

$$F(z)=f_1(z)\bar{z}+f_2(z) \qquad (1)$$