Materials of the Conferences

ABOUT THE THEOREM OF SOKHOTSKY-WEIERSTRASS TYPE FOR TRIANALYTIC FUNCTIONS

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The classification of isolated singular points [5-7] of poly-analytic functions [3-5, 10], appearing due to the theorem of Sokhotsky-Weierstrass type [1, 2-5, 10], does not exclude the possibility of its being worked out in detail for some particular kinds of these functions, for example, for those of them, the structure of limit sets in l-points and o-points of which can be specified. In particular, such a specification can be offered for tri-analytic functions [8, 9], as a theorem, which is a nontrivial specifying and adding to a corresponding theorem for poly-analytic functions, is possible to be established for them.

Theorem

A limit set in an arbitrary point $a \in \overline{\mathbf{C}}$ of any tri-analytic function

$$F(z) = f_0(z) + f_1(z)\bar{z} + f_2(z)\bar{z}^2, \qquad (1)$$

prescribed in some deleted neighborhood of this point, consists of a point (finite or infinite), or a line of the plane $\overline{\mathbf{C}}$, or grouping of two lines, or a half-line, or a parabola of the second order, or a circle, or a Pascal snail of any of the four kinds with the isolated point elimination, or finally, is total, i.e. coincides with the whole plane $\overline{\mathbf{C}}$; here and further the lines, half-lines and parabolas of the plane $\overline{\mathbf{C}}$ being considered as completed with ∞ point.

Vice versa, for a point $a \in \overline{\mathbf{C}}$ and a set $\Phi \subset \overline{\mathbf{C}}$, which is a line or a grouping of two arbitrary lines (parallel or concurrent), or a half-line, or a parabola of the second order, or a circle, or a Pascal snail of any of the four kinds with the isolated point elimination, or finally, for $\Phi = \{c\}$ with *c* as an arbitrary point of the plane $\overline{\mathbf{C}}$, a tri-analytic function $F(z)$ with analytical components – rational functions from the field $\mathbf{C}(z)$, for which $C(F(z),a) = \Phi$, exists.

Deduction

If the point $a \in \overline{\mathbf{C}}$ is an essentially singularity for at least one analytic function $f_k(z)$ $(k = 0,1,2)$, it is the only case when $C(F(z),a)$ is total, i.e. coincides with $\overline{\mathbf{C}}$ [3-4, 7].

Now let the point $a \in \overline{\mathbf{C}}$ be not an essential singularity for none of the analytic components $f_k(z)$ $(k = 0,1,2)$ of the tri-analytic function $F(z)$, but then a representation (3) of [10] takes place, if $a = \infty$, or an analog representation for the function $F(z) = F(z + a)$ in some deleted neighborhood of the null point. Now the structure of the limit set $C(F(z),a)$ will be fully defined by the corresponding functions $F_\infty(z)$ и $F_w(z)$ [5, 10] (the function $F_0(z)$ can be ignored, as $C(F_0(z),a) = \{0\}$).

If $F_\infty(z) \equiv 0$, and $F_w(z) \equiv c \in \mathbf{C}$, then $C(F(z),a) = \{c\}$; and if $F_\infty(z) \equiv 0$ and $F_w(z)$ is not identical to the constant, then $C(F(z),a) = P(w)$ where $w = \{z : |z| = 1\}$ is a unit circle and $P(z) = a_{(-2,2)}z^2 + a_{(-1,1)}z + a_{(0,0)}$ is a first or second degree polynomial [5, 9, 10] from the ring $\mathbf{C}[z]$. In the first case $P(w)$ is, obviously, a circle; and in the second one – a Pascal snail of any of the four kinds [11] or again a circle.

The given conclusion for the second case results from an easily fixed auxiliary fact.

Lemma

The image of the circle $w = \{z : |z| = 1\}$ with mapping given by a second degree polynomial $P(z) = Az^2 + Bz + C$ will be:

with $B = 0$ - a circle of radius $|A|$ and centre O;

with $B \neq 0$ - a Pascal snail (with its isolated point elimination), the snail having a juncture if $|B| < 2|A|$; the snail being a cardioid with a cuspidal point if $|B| = 2|A|$; the snail having the only point in which right and left tangents coincide (and the snail will have two flex points) if если $2|A| < |B| < 4|A|$; the snail having the only one tangent in its every point , but not having a cuspidal point if $|B| \geq 4|A|$.

Deduction

Taking $A \neq 0$ and $B \neq 0$, let us assume the polynomial $P(z) = Az^2 + Bz + C$, $A \neq 0$ as follows

$$P(z) = A\left(z + \frac{B}{2A}\right)^2 + C - \frac{B^2}{4A}.$$

Now let us write a nonvanishing number $\frac{B}{2A}$ in an exponential form $\frac{B}{2A} = re^{ij}$ with $r = \left|\frac{B}{2A}\right| > 0$, $0 \leq j < 2p$, and finally, reduce the polynomial to the form $P(z) = Ae^{2ij}(ze^{-ij} + r)^2 + C - Ar^2e^{2ij}$.Now, obviously, it is enough to clear up what is the image of the unit circle $w = \{z : |z| = 1\}$ with mapping given by the following auxiliary polynomial: $\tilde{p}(z) = (z + r)^2$, with $r > 0$; but if $z = e^{it}$ with $0 \leq t < 2p$, then

$$\tilde{p}(z) = (e^{it} + r)^2 = (\cos t + i\sin t + r)^2 =$$

$$= 2\cos^2 t + 2r\cos t + r^2 - 1 + 2i(r + \cos t)\sin t,$$

that means [11]: $\tilde{p}(w)$ is a Pascal snail (with the isolated point elimination) with the parameters $a = 2$ and $l = 2r$, which is biased to the vector $(r^2 - 1; 0)$. The lemma is proved.

Now let the function $F_\infty(z)$ be not identical to the constant. Then $C(F(z), a) = \{\infty\}$ or $C(F(z), a)$ is a join of a finite number of nontrivial polynomial lines augmented with ∞ point. The last situation for tri-analytic functions needs to be specified (1). To do it we have to change the function $F_w(z)$ with the corresponding constant [5, p. 58], then apply the described in [5] (theorem 3.1) poly-analytic property deflation mode of the function $F_\infty(z)$. The made transformations allow to make sure that the above join can be a line, or a grouping of two lines, or a half-line, or a parabola of the second order (with adding to every of these sets ∞ point).

As a supplement to the deduction of the first part of the theorem it is worth taking into account the first part of the theorem 3.2 from [5], and also the possibility to transform the case $a = 0$ to the case $a = \infty$ (and vice versa) by means of a simple change of the variables – changing symbol z into the expression $\frac{1}{z}$ and multiplying the given function by the fraction $\left(\frac{\bar{z}}{z}\right)^2$ with the further considering of this factor's possibility to converge to one of not more than two complex numbers ([5], p. 57).

The deduction of the second part of the theorem becomes easy when denoting concrete samples of realization of all the numerated in the theorem cases, the structuring of the corresponding samples being obvious or being contained in [5,6].

Note

It is obvious that the given theorem allows to offer a more delicate classification of isolated singularities of tri-analytic functions compared to the classification of these points for poly-analytic general functions, and namely, to excel, alongside with the essential singularity, removable (on continuity) singularity and a pole, four kinds of l- and o-points more.

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ABOUT SOME METHODS OF CONSTRUCTION POLYANALYTIC FUNCTIONS WITH A PREDETERMINED LIMIT SET IN THEIR ISOLATED SINGULAR L-POINT

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In this paper we consider a new method of quick construction of polyanalytic functions with a predetermined cluster set in isolated singular B-points (radiant points) of these functions.

It is known [1–5, 7] that if a set $M \subset \overline{\mathbf{C}}$ of any of the four types is given: an extended complex plane $\overline{\mathbf{C}}$, a polynomial image $P(w)$ of a unit circle $w = \{z \in \mathbf{C} : |z| = 1\}$ ($P(z) \in \mathbf{C}[z] \setminus \mathbf{C}$), an arbitrary unit subset of the set $\overline{\mathbf{C}}$, and finally, a union of finite number of nondegenerate polynomial lines $\bigcup_{j=1}^{m} L_j$ augmented with ∞ point, then for any point $a \in \overline{\mathbf{C}}$ there is such a deleted neighborhood $\overset{0}{O}(a)$ of its and a defined in it poly-analytic (p.a.) function $F(z)$, i.e. the function of the kind

$$F(z) = \sum_{k=0}^{n-1} f_k(z)\bar{z}^k, \qquad (1)$$

where $n \in N$, $f_k(z)$ ($k = 0, ..., n-1$) – are analytic in $\overset{0}{O}(a)$ functions, that $C(F(z), a) = M$. Let us remind that the number $n \in N$ is called [1] the order of poly-analytic property of the function $F(z)$, and if